Differential and integral operators on Appell's matrix functions

R. S. Batahan M. S. Metwally

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Abstract:

This paper deals with the study of the Appell's matrix functions. An integral representation of these functions is derived here. Furthermore, certain relations are obtained by acting some operators to the Appell's matrix functions.

1. **Introduction.**

Appell's functions of scalar coefficients and variables appear in many fields such as statistical distribution theory, heat flow, astrophysics and related areas see for instance Exton [3] and Srivastava and Karlsson [12]. Some properties of Appell's functions in particular the second Appell's function has been recently presented in [2].

Appell's functions of real variables were generalized to these functions with matrix argument in [8] and [10], see also the book of Mathai [9]. Recently, Upadhyaya and Dhami have presented some properties of the Appell's functions of matrix arguments in [13, 14]. Jódar and Cortés introduced and studied the hypergeometric matrix functions in [6, 7]. Some properties of gamma and beta matrix functions were given in [5].

Our main purpose in this paper is to obtain an extension of the hypergeometric matrix function to functions of more than one variable. Appell's matrix functions will be introduced as functions of two complex variables with matrix coefficients. The structure of this paper is the following:

Section 2 is organized to establish the four Appell's matrix functions and calculate its corresponding radius of regularity. In section 3 some integral representations for the first and the second of Appell's matrix functions are given. The functions contiguous to the

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Appell's matrix functions and its relations are given in section 4. In Section 5 we act some operators on the Appell's matrix functions.

A matrix P in $C^{N\times N}$ is a positive stable matrix if Re(λ)> 0 for all $\lambda \in \sigma(P)$ where $\sigma(P)$ is the set of all eigenvalues of P. Let P and Q be two positive stable matrices in $C^{N\times N}$. The Gamma matrix function, $\Gamma(P)$ and the Beta matrix function, $B(P,Q)$ have been defined in [5], as follows:

(1.1) $$\Gamma(P)=\int_0^\infty e^{-t}t^{P-I}dt;\ t^{P-I}=\exp((P-I)\ln t),$$

and

(1.2) $$B(P,Q)=\int_0^1 t^{P-I}(1-t)^{Q-I}\,dt,$$

where I is the identity matrix in $C^{N\times N}$.

If P+ nI is invertible for every integer n ≥ 0, then

(1.3) $$(P)_n = P(P+I)(P+2I)\ldots(P+(n-1)I);n\geq 1;\ (P)_0 = I,$$

and

(1.4) $$(P)_n = \Gamma(P+nI)\Gamma^{-1}(P);\quad n\geq 0.$$

From [5, theorem 1], we have

(1.5) $$\Gamma(P)=\lim_{n\to\infty}(n-1)!(P)_n^{-1}n^P.$$

Let P and Q be commuting matrices in $C^{N\times N}$ such that the matrices $P + nI$, $Q + nI$ and $P + Q + nI$ are invertible for every integer $n \geq 0$. Then according to [6, theorem 2] we have

(1.6) $$B(P,Q)=\Gamma(P)\Gamma(Q)\Gamma^{-1}(P+Q)\,.$$

The hypergeometric matrix function $F(A,B;C;z)$ was given in the form [6]

(1.7) $$F(A,B;C;z)=\sum_{n=0}^{\infty}\frac{(A)_n(B)_n\left[(C)_n\right]^{-1}}{n!}z^n\,,$$

for the matrices *A*, *B* and *C* in $C^{N\times N}$ such that *C*+ *nI* is invertible for all integer $n\geq 0$ and for $|z|<1$.

If *CB* = *BC* and *C*, *B*, *C* – *B* are positive stable matrices, then for $|z|<1$, the series (1.7) has an integral representation in the form [6, theorem 5]

(1.8)

$$F(A,B;C;z)=\left(\int_0^1(1-tz)^{-A}t^{B-I}(1-t)^{C-B-I}\,dt\right)\Gamma^{-1}(B)\Gamma^{-1}(C-B)\Gamma(C)$$

.

The Schur decomposition of *P* was given by [4, pp. 192 – 193] in the form:

(1.9) $$\left\|e^{tP}\right\|\leq e^{tM(P)}\sum_{k=0}^{r-1}\frac{\left(\|P\|r^{1/2}t\right)^k}{k!},\qquad t\geq 0\,.$$

2. **Definition of Appell's matrix functions.**

Let *A*, A_1, *B*, B_1, *C* and C_1 be matrices in $C^{N\times N}$ such that *C*+*nI* and C_1+nI are invertible for every integer $n\geq 0$.

We define Appell's matrix functions as follows:

(2.1)

$$F_1(A;B,B_1;C;z,w)=\sum_{m,n=0}^{\infty}\tfrac{1}{m!n!}(A)_{m+n}(B)_m(B_1)_n\left[(C)_{m+n}\right]^{-1}z^m w^n\,,$$

(2.2)

$$F_2(A;B,B_1;C,C_1;z,w)=\sum_{m,n=0}^{\infty}\frac{1}{m!n!}(A)_{m+n}(B)_m(B_1)_n\left[(C)_m\right]^{-1}\left[(C_1)_n\right]^{-1}z^m w^n$$

,

(2.3)

$$F_3(A,A_1;B,B_1;C;z,w)=\sum_{m,n=0}^{\infty}\frac{1}{m!n!}(A)_m(A_1)_n(B)_m(B_1)_n\left[(C)_{m+n}\right]^{-1}z^m w^n$$

,

and

(2.4)

$$F_4(A,A_1;B;C,C_1;z,w)=\sum_{m,n=0}^{\infty}\frac{1}{m!n!}(A)_m(A_1)_n(B)_{m+n}\left[(C)_m\right]^{-1}\left[(C_1)_n\right]^{-1}z^m w^n$$

,

where for any matrix P in $C^{N\times N}$ $(P)_s$ as defined in (1.3).

Firstly, let us state and prove the following:

Lemma (2.1):

The matrix function $F(z,w)=\sum_{m,n=0}^{\infty}A_{m,n}z^m w^n$ is absolutely and uniformly convergent in the open hypersphere $S_R; R>0$ if

(2.5) $$\overline{\lim_{m+n\to\infty}}\left\{\frac{\|A_{m,n}\|}{\sigma_{m,n}}\right\}^{1/(m+n)}=\frac{1}{R}<\infty,$$

where

$$\sigma_{m,n} = \begin{cases} \dfrac{(m+n)^{(m+n)/2}}{m^{m/2} n^{n/2}} & m,n > 0; \\ 1 & m = n = 0. \end{cases}$$

Proof: We say that the matrix function $F(z,w)$ is convergent if $\|F(z,w)\|$ converges. That is we will investigate the convergence of $\|F(z,w)\|$. Let R_1 be any positive number such that $R_1 < R$, and let us consider the series

(2.6) $$\|F(z,w)\| \leq \sum_{m,n=0}^{\infty} \|U_{m,n}\|,$$

where $\|U_{m,n}\| = \|A_{m,n}\| \left\{ \max_{\bar{S}_{R_1}} |z^m w^n| \right\}.$

We will exploit the relation due to Sayyed [11], in the form:

$$\max_{\bar{S}_{R_1}} |z^m w^n| = \frac{R_1^{m+n}}{\sigma_{m,n}};\ m,\ n \geq 0.$$

Thus, it follows

$$\|U_{m,n}\| = \|A_{m,n}\| \frac{R_1^{m+n}}{\sigma_{m,n}}.$$

Hence (2.5) gives

(2.7) $$\overline{\lim_{m+n\to\infty}} \|U_{m,n}\|^{1/(m+n)} = \frac{R_1}{R} < 1.$$

If R_2 is a real number such that $R_1 < R_2 < R$, then, from (2.7), there exists an integer m_0 such that

$$\|U_{m,n}\| < \left(\frac{R_2}{R}\right)^{m+n}; \quad m+n > m_0.$$

Clearly, one finds

$$\sum_{m,n=0}^{\infty}\left\|U_{m,n}\right\| < K + \sum_{m>m_0}(m+1)\left(\frac{R_2}{R}\right)^m$$

$$< K + \left(1-\frac{R_2}{R}\right)^{-2} < \infty ,$$

where K denotes positive finite numbers independent of m and n, and which does not retain the same values at different occurrences, here

$$K = \sum_{0\le m+n\le m_0}^{\infty}\left\|U_{m,n}\right\| < \infty$$

Therefore the series (2.6) is absolutely and uniformly convergent in S_{R_1}.

Since R_1 can be chosen arbitrary near to R, then the lemma is proved.

Now, let us the first Appell's matrix function F_1 by

(2.8) $$F_1(A;B,B_1;C;z,w) = \sum_{m,n=0}^{\infty} M_{m,n} z^m w^n ,$$

where $M_{m,n} = \frac{1}{m!n!}(A)_{m+n}(B)_m(B_1)_n\left[(C)_{m+n}\right]^{-1}$.

Thus

$$\left\|F_1(A;B,B_1;C;z,w)\right\| \le \sum_{m,n=0}^{\infty}\left\|M_{m,n}\right\|\left|z\right|^m\left|w\right|^n ,$$

then, by using lemma (2.1) we have

$$R = \overline{\lim_{m+n\to\infty}}\left\{\frac{\left\|U_{m,n}\right\|}{\sigma_{m,n}}\right\}^{1/(m+n)} ,$$

$$= \overline{\lim_{m+n\to\infty}}\left\{\frac{1}{m!n!\sigma_{m,n}}\left\|(A)_{m+n}(B)_m(B_1)_n\left[(C)_{m+n}\right]^{-1}\right\|\right\}^{1/(m+n)} .$$

Since $1 \leq \sigma_{m,n} \leq \left(\sqrt{2}\right)^{m+n}$, then with the help of (1.5) it follows

$$R \leq \overline{\lim_{m+n\to\infty}} \left\{ \left\| \frac{(m+n)^{-A}(A)_{m+n}}{(m+n-1)!} \right\| \left\|(m+n)^{A}\right\| \left\| \frac{m^{-B}(B)_m}{(m-1)!} \right\| \left\| m^{B} \right\| \left\| \frac{n^{-B_1}(B_1)_n}{(n-1)!} \right\| \left\| n \right.\right.$$

$$\left. \left\| (m+n-1)! \left[(C)_{m+n} \right]^{-1} (m+n)^{C} \right\| \left\| (m+n)^{-C} \right\| \frac{1}{mn\sigma_{m,n}} \right\}^{1/(m+n)}$$

$$\leq \overline{\lim_{m+n\to\infty}} \left\{ \left\| \Gamma^{-1}(A) \right\| \left\| \Gamma^{-1}(B) \right\| \left\| \Gamma^{-1}(B_1) \right\| \left\| \Gamma^{-1}(C) \right\| \right\}^{1/(m+n)}$$

$$\times \overline{\lim_{m+n\to\infty}} \left\{ \left\| (m+n)^{A-C} \right\| \left\| m^{B-I} \right\| \left\| n^{B_1-I} \right\| \right\}^{1/(m+n)}$$

$$= \overline{\lim_{n(\alpha+1)\to\infty}} \left\{ \left\| (n(\alpha+1))^{A-C} \right\| \left\| (\alpha n)^{B-I} \right\| \left\| n^{B_1-I} \right\| \right\}^{1/(n(\alpha+1))}$$

$$= \overline{\lim_{n(\alpha+1)\to\infty}} \left\{ \left\| (\alpha+1)^{A-C} \right\| \left\| \alpha^{B-I} \right\| \right\}^{1/(n(\alpha+1))} \left\{ \left\| n^{A-C} \right\| \left\| n^{B-I} \right\| \left\| n^{B_1-I} \right\| \right\}^{1/(n(\alpha+1))}$$

.

By using (1.7) we can write

$$\left\| n^{A-C} \right\| \leq n^{M(A-C)} \sum_{k=0}^{r-1} \frac{\left(\|A-C\| r^{1/2} \ln n \right)^k}{k!},$$

$$\left\| n^{B-I} \right\| \leq n^{M(B)-1} \sum_{k=0}^{r-1} \frac{\left(\|B-I\| r^{1/2} \ln n \right)^k}{k!},$$

and

$$\left\| n^{B_1 - I} \right\| \le n^{M(B_1) - 1} \sum_{k=0}^{r-1} \frac{\left(\|B_1 - I\| r^{1/2} \ln n \right)^k}{k!}.$$

Thus, we have

$$R \le \overline{\lim_{n(\alpha+1)\to\infty}} \left\{ n^{M(A-C)+M(B)+M(B_1)-2} \right\}^{1/(n(\alpha+1))}$$

$$\overline{\lim_{n(\alpha+1)\to\infty}} \left\{ \left[\sum_{k=0}^{r-1} \frac{\left(\|A-C\| r^{1/2} \ln n \right)^k}{k!} \right] \left[\sum_{k=0}^{r-1} \frac{\left(\|B-I\| r^{1/2} \ln n \right)^k}{k!} \right] \right.$$

$$\left. \left[\sum_{k=0}^{r-1} \frac{\left(\|B_1 - I\| r^{1/2} \ln n \right)^k}{k!} \right] \right\}^{1/(n(\alpha+1))} .$$

Since,

$$\sum_{k=0}^{r-1} \frac{\left(\|P\| r^{1/2} \ln n \right)^k}{k!} \le (r \ln n)^{r-1} \sum_{k=0}^{r-1} \frac{(\|P\|)^k}{k!}$$

$$\le (r \ln n)^{r-1} \sum_{k=0}^{\infty} \frac{(\|P\|)^k}{k!} = (r \ln n)^{r-1} e^{\|P\|},$$

for every matrix P in $C^{N\times N}$, then

$$R \le \overline{\lim_{n(\alpha+1)\to\infty}} \left\{ (r \ln n)^{3(r-1)} e^{\|A-C\|+\|B-I\|+\|B_1-I\|} \right\}^{1/(n(\alpha+1))} = 1$$

.

Therefore, the unity is the radius of regularity of the first Appell's matrix function F_1 and so the first Appell's matrix function F_1 is regular in the hypersphere $\bar{S}_r; r = 1$.

By analogous way, one can prove that the other Appell's matrix function F_2, F_3 and F_4 are regular in the hypersphere $\bar{S}_r; r = 1$.

3. Integral representations.

Let A, B, B_1 and C be matrices in $C^{N\times N}$ such that

(3.1) $AB = BA$ and $B_1C = C B_1$,

and $C+nI$ are invertible for every integer $n \geq 0$.
Suppose that

(3.2) $AC = CA$,

(3.3) A, C and $C - A$ are positive stable.

By (1. 4) we can write

$$(A)_{m+n}\left[(C)_{m+n}\right]^{-1} = \Gamma^{-1}(A)\Gamma(A+(n+m)I)\Gamma^{-1}(C+(n+m)I)\Gamma(C)$$

$$= \Gamma^{-1}(A)\Gamma^{-1}(C-A)\Gamma(C-A)$$

$$\Gamma(A+(n+m)I)\Gamma^{-1}(C+(n+m)I)\Gamma(C).$$

By (3.2) and (3.3) with the help of (1.2) and (1.6) we have

$$\Gamma(C-A)\Gamma(A+(n+m)I)\Gamma^{-1}(C+(n+m)I) = \int_0^1 t^{A+(m+n-1)I}(1-t)^{C-A-I}\,dt$$

.

Hence

(3. 4) $$(A)_{m+n}\left[(C)_{m+n}\right]^{-1} = \Gamma^{-1}(A)\Gamma^{-1}(C-A)$$

$$\left(\int_0^1 t^{A+(m+n-1)I}(1-t)^{C-A-I}\,dt\right)\Gamma(C).$$

From (2.1) and by (3. 1) we can write

$$F_1(A;B,B_1;C;z,w)=\sum_{m,n=0}^{\infty}\frac{1}{m!n!}(B)_m(A)_{m+n}\left[(C)_{m+n}\right]^{-1}(B_1)_n z^m w^n$$

$$=\sum_{m,n=0}^{\infty}\frac{(B)_m}{m!}\Gamma^{-1}(A)\Gamma^{-1}(C-A)\left(\int_0^1 t^{A(m+n-1)I}(1-t)^{C-A-I}\,dt\right)\Gamma(C)\frac{(B_1)_n}{n!}z^m w^n$$

.

Since the series and the integral can be permuted in the above expression, so we have:

$$F_1(A;B,B_1;C;z,w)=\int_0^1\sum_{m=0}^{\infty}\frac{(B)_m}{m!}z^m t^{A(m+n-1)I}(1-t)^{C-A-I}$$

$$\sum_{n=0}^{\infty}\frac{(B_1)_n}{n!}w^n dt\Gamma^{-1}(A)\Gamma^{-1}(C-A)\Gamma(C)$$

$$=\int_0^1\left\{\sum_{m=0}^{\infty}\frac{(B)_m}{m!}(tz)^m\right\}t^{A-I}(1-t)^{C-A-I}\left\{\sum_{n=0}^{\infty}\frac{(B_1)_n}{n!}(tw)^n\right\}dt$$

$$\times\Gamma^{-1}(A)\Gamma^{-1}(C-A)\Gamma(C)$$

$$=\left(\int_0^1(1-tz)^{-B}(1-tw)^{-B_1}t^{A-I}(1-t)^{C-A-I}\,dt\right)\Gamma^{-1}(A)\Gamma^{-1}(C-A)\Gamma(C)$$

.

Therefore, we state the following:

Theorem (3.1):

Let A, B, B_1 and C be matrices in $C^{N\times N}$ satisfy (3.1), (3.2) and (3.3) and C+nI are invertible for every integer n ≥ 0. Then the first Appell's matrix function has the following integral representation:

(3.5) $$F_1(A;B,B_1;C;z,w)=$$

$$\left(\int_0^1 (1-tz)^{-B} (1-tw)^{-B_1} t^{A-I} (1-t)^{C-A-I}\, dt \right) \Gamma^{-1}(A) \Gamma^{-1}(C-A) \Gamma(C)$$

.

Now, we consider the second Appell's matrix function F_2. Suppose that

(3.6) $B_1 C = C B_1$,

(3.7) $B_1 C_1 = C_1 B_1$,

and

(3.8) B_1, C_1 and $B_1 - C_1$ are positive stable.

Since

$$(A)_{m+n} = (A)_m (A+mI)_n ,$$

then by (3.6), (2.2) becomes

(3.9)
$$F_2(A; B, B_1; C, C_1; z, w)$$

$$= \sum_{m,n=0}^{\infty} \frac{(A)_m (A+mI)_n (B)_m}{m!\, n!} \left[(C)_m\right]^{-1} (B_1)_n \left[(C_1)_m\right]^{-1} z^m w^n$$

.

Similarly to (3.4) and by using (3.7) and (3.8) we get

$$(B_1)_n \left[(C_1)_n\right]^{-1} = \Gamma^{-1}(B_1) \Gamma^{-1}(C_1 - B_1)$$

$$\left(\int_0^1 t^{B_1 + (n-1)I} (1-t)^{C_1 - B_1 - I}\, dt \right) \Gamma(C_1).$$

Thus, (3.9) becomes

$$F_2(A;B,B_1;C,C_1;z,w)=\sum_{m,n=0}^{\infty}\frac{(A)_m(A+mI)_n(B)_m}{m!\,n!}\left[(C)_m\right]^{-1}$$

$$\times\left(\int_0^1 t^{B_1+(n-1)I}(1-t)^{C_1-B_1-I}\,dt\right)z^m w^n\Gamma^{-1}(B_1)\Gamma^{-1}(C_1-B_1)\Gamma(C_1)$$

$$=\int_0^1\sum_{m=0}^{\infty}\left\{\sum_{n=0}^{\infty}\frac{(A+mI)_n}{n!}(tw)^n\right\}\frac{(A)_m(B)_m}{m!}\left[(C)_m\right]^{-1}z^m$$

$$\times t^{B_1-I}(1-t)^{C_1-B_1-I}\,dt\quad\Gamma^{-1}(B_1)\Gamma^{-1}(C_1-B_1)\Gamma(C_1)$$

$$=\int_0^1(1-tw)^{-A}\sum_{m=0}^{\infty}\frac{(A)_m(B)_m}{m!}\left[(C)_m\right]^{-1}\left(\frac{z}{1-tw}\right)^m$$

$$\times t^{B_1-I}(1-t)^{C_1-B_1-I}\,dt\quad\Gamma^{-1}(B_1)\Gamma^{-1}(C_1-B_1)\Gamma(C_1).$$

Therefore, the following result has been established:

Theorem (3.2):

Let A, B, B_1, C and C_1 be matrices in $C^{N\times N}$ *satisfy* (*3.6*), (*3.7*) *and* (*3.8*) *where C+nI and C_1+nI are invertible for every integer n ≥ 0. Then the second Appell's matrix function has the following integral representation:*

(3.10) $$F_2(A;B,B_1;C,C_1;z,w)=$$

$$\left\{\int_0^1(1-tw)^{-A}F\left(A,B;C;\frac{z}{1-tw}\right)t^{B_1-I}(1-t)^{C_1-B_1-I}\,dt\right\}$$

$$\times\Gamma^{-1}(B_1)\Gamma^{-1}(C_1-B_1)\Gamma(C_1).$$

4. The contiguous function relations.

In this section we give the contiguous function relations concerning to the first Appell's matrix function. We use the notation

(4.1) $$F_1 = \sum_{m,n=0}^{\infty} U_{m,n}(z,w),$$

where $U_{m,n}(z,w) = \frac{1}{m!n!}(A)_{m+n}(B)_m(B_1)_n\left[(C)_{m+n}\right]^{-1} z^m w^n$.

Clearly, the multiplication in $(P)_n$ is commutative for every P in $C^{N\times N}$.

The contiguous function relations to $F_1(A,B,B_1;C,C_1;z,w)$ may be written in the form

(4.2) $$F_1(A+) = \sum_{m,n} A^{-1}(A+(m+n)I)\ U_{m,n}(z,w),$$

(4.3) $$F_1(A-) = \sum_{m,n} (A-I)(A-I+(m+n)I)^{-1} U_{m,n}(z,w).$$

If

(4.4) $A\,B = B\,A$,

then we can write

(4.5) $$F_1(B+) = \sum_{m,n} B^{-1}(B+mI)\ U_{m,n}(z,w),$$

(4.6) $$F_1(B-) = \sum_{m,n} (B-I)(B-I+mI)^{-1} U_{m,n}(z,w).$$

Also, if

(4.7) $B_1\, C = C\, B_1$,

then it follows

(4.8) $$F_1(B_1+) = \sum_{m,n} U_{m,n}(z,w)(B_1+nI)B_1^{-1},$$

(4.9) $$F_1(B_1-) = \sum_{m,n} U_{m,n}(z,w)(B_1 - I + nI)^{-1}(B_1 - I).$$

Finally, we give

(4.10) $$F_1(C+) = \sum_{m,n} U_{m,n}(z,w)(C + (m+n)I)^{-1}C\,,$$

(4.11) $$F_1(C-) = \sum_{m,n} U_{m,n}(z,w)(C - I + (m+n)I)(C - I)^{-1}.$$

Now, consider the differential operator D in the form

(4.12) $D = d_1 + d_1\,,$

where $d_1 = z\dfrac{\partial}{\partial z}$ and $d_2 = w\dfrac{\partial}{\partial w}$.

It follows

(4.13) $(D \blacksquare\blacksquare A)F_1 = AF_1(A+)\,,$

(4.14) $DF_1 + BF_1 + F_1B_1 = BF_1(B+) + F_1(B_1+)B_1\,,$

and

(4.15) $DF_1 + F_1(C - I) = F_1(C-)(C - I).$

From (4.13), (4.14) and (4.15), we get

(4.16) $(A - B)F_1 - F_1B_1 = AF_1(A+) - BF_1(B+) - F_1(B_1+)B_1\,,$

and

(4.17) $AF_1 - F_1(C - I) = AF_1(A+) - F_1(C-)(C - I).$

5. Appell's matrix functions under some operators.

This section is intended for investigate the acting of some operators on the first Appell's matrix function. Firstly, operate with d_1 on the function $F_1(A, B, B_1; C; z, w)$ to get

$$d_1F_1(A, B, B_1; C; z, w)$$

$$= \sum_{m=1,n=0}^{\infty} \frac{m(A)_{m+n}(B)_m(B_1)_n\left[(C)_{m+n}\right]^{-1}}{m!\,n!} z^m w^n$$

$$= \sum_{m,n=0}^{\infty} \frac{(A)_{m+1+n}(B)_{m+1}(B_1)_n\left[(C)_{m+1+n}\right]^{-1}}{m!\,n!} z^{m+1} w^n .$$

Note that, by (1.3), $(P)_{n+1} = P(P+I)_n$, for every P in $C^{N\times N}$.
Thus,

$$d_1 F_1(A, B, B_1; C; z, w)$$
$$= z \sum_{m,n=0}^{\infty} \frac{A(A+I)_{m+n} B(B+I)_m (B_1)_n \left[(C+I)_{m+n}\right]^{-1} C^{-1}}{m!\,n!} z^m w^n$$

,

which can be written by (4.4) in the form
(5.1)

$$d_1 F_1(A, B, B_1; C; z, w) = zABF_1(A+I, B+I, B_1; C+I; z, w)C^{-1}.$$

Now,

$$d_1^2 F_1(A, B, B_1; C; z, w) = AB \sum_{m,n=0}^{\infty} \frac{(m+1)(A+I)_{m+n}(B+I)_m}{m!\,n!}$$

$$(B_1)_n \left[(C+I)_{m+n}\right]^{-1} z^{m+1} w^n C^{-1}$$

$$= AB \sum_{m,n=0}^{\infty} \frac{(A+I)_{m+1+n}(B+I)_{m+1}(B_1)_n \left[(C+I)_{m+1+n}\right]^{-1}}{m!\,n!} z^{m+2} w^n C^{-1}$$

$$AB \sum_{m,n=0}^{\infty} \frac{(A+I)_{m+n}(B+I)_m (B_1)_n \left[(C+I)_{m+n}\right]^{-1}}{m!\,n!} z^{m+1} w^n C^{-1}$$

$$= z^2(A)_2(B)_2 \sum_{m,n=0}^{\infty} \frac{(A+2I)_{m+n}(B+2I)_m(B_1)_n\left[(C+2I)_{m+n}\right]^{-1}}{m!n!} z^m w^n \left[(C)_2\right]^{-1} +$$

$$zAB \sum_{m,n=0}^{\infty} \frac{(A+I)_{m+n}(B+I)_m(B_1)_n}{m!n!}\left[(C+I)_{m+n}\right]^{-1} z^{m+1} w^n C^{-1}$$

.

Similarly, when operating with d_2 on the function $F_1(A,B,B_1;C;z,w)$ and by (4.7) we get

(5.2)

$$d_2 F_1(A,B,B_1;C;z,w) = wAF_1(A+I,B,B_1+I;C+I;z,w)B_1C^{-1}.$$

Therefore, with (4.4) and (4.7) we obtain

(5.3)

$$DF_1(A,B,B_1;C;z,w) = zABF_1(A+I,B+I,B_1;C+I;z,w)C^{-1} +$$

$$wAF_1(A+I,B,B_1+I;C+I;z,w)B_1C^{-1}.$$

Now, from (4.15), we have

(5.4) $$DF_1(A,B,B_1;C;z,w) = \left[F_1(A,B,B_1;C-I;z,w) - F_1(A,B,B_1;C;z,w)\right](C-I)$$

,

and

(5.5)

$$D^2F_1(A,B,B_1;C;z,w) = F_1(A,B,B_1;C-2I;z,w)(C-2I)(C-I) - F_1(A,B,B_1;C-I;z,w)\left\{(C-2I)(C-I)+(C-I)^2\right\} + F_1(A,B,B_1;C;z,w)(C-I)^2.$$

Hence, by mathematical induction, we obtain the following general form:

(5.6) $$D^n F_1(A,B,B_1;C;z,w) = F_1(A,B,B_1;C-nI;z,w)$$

$$\prod_{k=1}^{n}(C-kI) -$$

$$F_1(A,B,B_1;C-(n-1)I;z,w)\left\{\prod_{k=1}^{n}(C-kI) + \prod_{k=1}^{n-1}(C-kI)\sum_{k=1}^{n-1}(C-kI)\right\} +$$

$$F_1(A,B,B_1;C-(n-2)I;z,w)\left\{\prod_{k=1}^{n-1}(C-kI)\sum_{k=1}^{n-1}(C-kI) + \right.$$

$$\left.\prod_{k=1}^{n-2}(C-kI)\left[\sum_{k=1}^{n-2}(C-kI)\right]^2\right\} + \cdots + (-1)^n F_1(A,B,B_1;C;z,w)(C-I)^n$$

.

Abul-Ez and Sayyed [1] introduced the integral operator in the form:

(5.7) $$I_{zw} = \begin{cases} 1 & ; \quad m,n=0, \\ I_z + I_w & ; \quad otherwise. \end{cases}$$

where $I_z = \frac{1}{z}\int_0^z dz$ and $I_w = \frac{1}{w}\int_0^w dw$.

We act the operator I_z on the first Appell's matrix function to obtain

$$I_z F_1(A,B,B_1;C;z,w) = \sum_{m,n=0}^{\infty} \frac{(A)_{m+n}(B)_m(B_1)_n\left[(C)_{m+n}\right]^{-1}}{(m+1)!\,n!} z^m w^n$$

$$= \sum_{m=1,n=0}^{\infty} \frac{(A)_{m-1+n}(B)_{m-1}(B_1)_n\left[(C)_{m-1+n}\right]^{-1}}{m!\,n!} z^{m-1} w^n$$

Clearly, from (1.3) one finds that for every matrix P in $C^{N\times N}$ the following holds

(5.8) $$(P)_{n-1} = (P-I)^{-1}(P-I)_n.$$

Therefore, by using (4.4) and (5.8) we have

(5.9) $$I_z F_1(A,B,B_1;C;z,w) = \frac{1}{z}(A-I)^{-1}(B-I)^{-1}$$

$$\sum_{m=1,n=0}^{\infty} \frac{(A-I)_{m+n}(B-I)_m(B_1)_n\left[(C-I)_{m+n}\right]^{-1}}{m!n!} z^m w^n (C-I)$$

$$= -\frac{1}{z}(A-I)^{-1}(B-I)^{-1}(C-I) + \frac{1}{z}(A-I)^{-1}(B-I)^{-1}$$

$$\sum_{m,n=0}^{\infty} \frac{(A-I)_{m+n}(B-I)_m(B_1)_n\left[(C-I)_{m+n}\right]^{-1}}{m!n!} z^m w^n (C-I)$$

$$= -\frac{1}{z}(A-I)^{-1}(B-I)^{-1}(C-I) + \frac{1}{z}(A-I)^{-1}(B-I)^{-1}$$

$$F_1(A-I;B-I,B_1;C-I;z,w)(C-I).$$

Similarly when acting the operator I_w on the first Appell's matrix function by using (4.7) we find

(5.10)

$$I_w F_1(A,B,B_1;C;z,w) = -\tfrac{1}{w}(A-I)^{-1}(B_1-I)^{-1}(C-I) + \tfrac{1}{w}(A-I)^{-1}$$

$$F_1(A-I;B,B_1-I;C-I;z,w)(B_1-I)^{-1}(C-I).$$

From (5.9) and (5.10) under the conditions (4.4) and (4.7) we obtain

$$I_{zw} F_1(A,B,B_1;C;z,w) =$$

$$-\frac{1}{z}(A-I)^{-1}(B-I)^{-1}(C-I) - \frac{1}{w}(A-I)^{-1}(B_1-I)^{-1}(C-I) +$$

$$\frac{1}{z}(A-I)^{-1}(B-I)^{-1}F_1(A-I;B-I,B_1;C-I;z,w)(C-I) +$$

$$\frac{1}{w}(A-I)^{-1}F_1(A-I;B,B_1-I;C-I;z,w)(B_1-I)^{-1}(C-I).$$

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