On Composite n dividing φ(n) τ(n)+2

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Abstract :

Let φ denote the Eular's Totitient . For any positive integer n let $\tau(n)$ denote the number of its positive divisors . If T is the set of all composite numbers n > 4 for which n divides $\varphi(n)\,\tau(n)+2$, we prove that every $n \in T$ has at least five distinct prime factors . Our result improves that of Yong-Gao and Jin –Hui Fang [4] and of Subbarao [3] .The proofs presented in the paper entirely different from the earlier authors .

Key words : Eular Totient function .

الملخص :

إذا كانت$\tau(n)$ عدد القواسم الموجبة للعدد الصحيح الموجب n و كانت φ دالة اويلر ، وإذا كانت T مجموعة الأعداد غير الأولية (المركبة) $4 < n$ بحيث أن العدد n يقسم المقدار $\varphi(n)\tau(n) + 2$ ، سنثبت في هذا العمل أنه إذا كان $n \in T$ فإن له على الأقل خمسة من المعاملات الأولية المختلفة ، النتائج التي حصل عليها الباحث هي تطوير لما حصل عليه الباحث سابورو [3] و الباحثان ينق – قو شن و جان – هيو فانج [4] . كما أن الإثباتات في هذا البحث تختلف عما قدمه الباحثون المذكورون .

المفتاح : دالة اويلر .

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1. Introduction

Let φ denotes is Euler totient function . For any positive integer n let $\tau(n)$ denote the number of positive divisors of n. Clearly n divides $\varphi(n)\tau(n)+2$ if n is prime or $n=4$ .Is this true for any composite n other than 4? That is, if

(1.1) $T_k=\{n:\ \varphi(n)\tau(n)+2=k\,n\}\ for\ k=1,2,3,\ldots$ and $T=\bigcup_{k=1}^{\infty}T_k$

Then the question seeks composite $n>4$ in T .Posing this problem in ([2], B37) it is recorded that Jud McCranie has shown that

(1.2) There is no composite $n\in T$ with $4<n<10^{10}$.

It is easy to see that every $n\in T$ is a squarefree so that it can be written as

(1.3) $n=p_1p_2p_3\ldots p_r\ with\ p_{1<}p_{2<}p_3\ldots<p_r$,where $\omega(n)=r$ is the number of distinct prime factors of n.

Yang-Gao Chen and Jin-Hui Fang[4] have shown that if $n\in T$ is of the form (1.3) then $p_i<(r\,2^{r-1})^{2^{i-1}}$ for $1\le i\le r$ .and remarked (see[4],Remark p.1) that using this inequalities they could prove that

(1.4) $\omega(n)\ge 5$ for $n\in T$.

1974, M.V.Subbarao [3] considered the same problem and proved that

1.5 . Theorem ([3], Theorem3).Any composite $n\in T$ with $n>4$ must have at least four distinct odd prime factors.

In fact he claimed that numerous computations were made (the details of which were not given in that paper) to prove Theorem 1.5.Also he showed the following:

1.6 Theorem ([3], Theorem4)

i) $T_k=\emptyset\ if\ k=1\ or\ 3\le k\le 14.$

ii) $T_2=\{4\}\cup P$ where P the set of all primes .

iii) If $n\in T_{15}$ then $\omega(n)=4\ or\ 5.$

Further he gave the possible values for $\omega(n)$ for any $n\in T_k$ with $16\le k\le 1024$.

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The purpose of this paper to give an entirely different proofs for (1.4). and Theorem 1.5; and to improve both of them and also 1.6 (iii). In fact , we prove

1.7 Theorem. Any composite $n \in T$ with $n > 4$ we must have at least five distinct odd prime factors .

In the process we show that $T_{15} = \emptyset$, improving (iii) of Theorem 1.6 ; and also give alternate prove of (i) of the same theorem .

2. Preliminary results

Let n denote the composite number in T with $n > 4$ and is the form (1.3) .Then $\tau(n) = 2^r$ so that $\varphi(n)\tau(n) + 2$ is of the form $2M$ where M is odd from which the following are immediate:

(2.1) if $n \in T_k$ then k and n are of opposite parity and that the even number of them is not divisible by 4 .

and

(2.2) $T_k = \emptyset$ if $4 \mid k$ .that is , $T_4 = T_8 = T_{12} = \cdots = \emptyset$.

In the rest of the paper , unless otherwise mentioned , n always composite number in T with $n > 4$ and is of the form (1.3) so that

(2.3) $(p_1 - 1)(p_2 - 1) \ldots (p_r - 1)2^r + 2 = kp_1p_2p_3 \ldots p_r$ for some $k \geq 1$.

2.4 Lemma. If $n \in T_k$ with $\omega(n) = r$ then $k < 2^{r-1}$ or $k < 2^r$ according as n is even or odd

Proof : Suppose $n \in T_k$ is even so that $p_1 = 2$. if $k \geq 2^{r-1}$ then it follows from (2.3) that

$2^{r-1}\{p_2p_3 \ldots p_r - (p_2 - 1)(p_3 - 1) \ldots (p_r - 1) \leq 1\}$, a contradiction .

If $n \in T_k$ is odd and if $k \geq 2^r$ then (2.3) gives the inequality

$2^r\{p_1p_2 \ldots p_r - (p_1 - 1)(p_2 - 1) \ldots (p_r - 1) \leq 2\}$, a contradiction since $r > 1$. Thus the lemma holds.

Let $\{q_i\}_{i=1}^{\infty}$ be the sequence of prime numbers in increasing order . that is , $q_1 = 2$, $q_2 = 3$, $q_3 = 5, q_4 = 7, \ldots$ Let

(2.5) $Q(r) = \prod_{i=1}^{r} \frac{q_{i+1}}{q_{i+1}-1}$, for $r = 1,2,3, \ldots$

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Note that $Q(1) = \frac{3}{2}, Q(2) = \frac{15}{8}, Q(3) = \frac{35}{16}, \ldots$ by simple induction on r the following inequalities can be proved

(2.6) $Q(r) < 2^r \ for\ r \geq 1.$

(2.7) $Q(r) \leq 2^{r-1}\ for\ r \geq 2.$

(2.8) $Q(r) \leq \frac{2^{r-1}}{3}\ for\ r \geq 4.$

We need the following well-know simple result :

(2.9) $\frac{x}{x-1}$ is a decreasing function for x > 1

For $n \in T_k$, we get from (2.3)that

(2.10) $P_n \doteq \frac{n}{\varphi(n)} = \prod_{i=1}^{r}\left(\frac{p_i}{p_i-1}\right) > \frac{2^r}{k}$.

If n is even then $p_i \geq q_i$ for $1 \leq i \leq r$ so that, by (2.9) , we have $P_n \leq \prod_{i=1}^{r}\frac{q_i}{q_i-1} = 2Q(r-1)$; while if n is odd then $p_i \geq q_{i+1}$for $1 \leq i \leq r$ so that again by (2.9), $P_n \leq \prod_{i=1}^{r}\frac{q_i}{q_i-1} = Q(r)$. Thus

(2.11) $P_n \leq \begin{cases} 2Q(r-1) & if\ n\ is\ even \\ Q(r) & if\ n\ is\ odd \end{cases}$

Now combining (2.10) and (2.11) we have , for $n \in T_k$ with $\omega(n) = r$ that

(2.12) $\frac{2^r}{k} < P_n \leq \begin{cases} 2Q(r-1) \text{ if } n \text{ is even} \\ Q(r) \text{ if } n \text{ is odd} \end{cases}$

2.13 Lemma. (i) $T_1 = \emptyset$. (ii) $T_2 = \{4\} \cup \{q_i : i \geq 1\}$.and (iii) $T_3 = \emptyset$.

Proof : (i) If possible $n \in T_1$, then (2.1) gives n is even and therefore by (2.12), $2^r < P_n \leq 2.Q(r-1)$ showing $Q(r-1) > 2^{r-1}$, a contradiction to (2.6). Hence $T_1 = \emptyset$.

(ii) Clearly $\{4\} \cup \{q_i : i \geq 1\} \subseteq T_2$. Conversely , if $n \in T_2$ and $n \neq 4$.Then n is odd and therefore by (2.12) , $Q(r) > 2^{r-1}$, which holds only for $r = 1$ in view of (2.7) showing n is a prime that is , $n = q_i$ for some $i \geq 1$. Thus $T_2 \subseteq \{4\} \cup \{q_i : i \geq 1\}$.

(iii) If possible , $n \in T_3$. Then $p_1 = 2$ and $r \geq 2$ so that by(2.12) we get $Q(r-1) > \frac{2^{r-1}}{3}$ which holds, in view of (2.8) only for $r = 2$ or 3. But when $r = 2$ the equation (2.3) with $k = 3$ gives $(p_2 - 1)4 + 2 = 6p_2$ which has no solution in prime p_2 ; and when $r = 3$ the equation (2.3) with $k = 3$ reduces

(2.14) $p_2p_3 + 5 = 4(p_2 + p_3)$

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So that $\frac{1}{p_2}+\frac{1}{p_3}>\frac{1}{4}$ which is impossible if $p_2 \geq 7$ and therefore $p_2 = 3$ or 5 ; and in either case there is no prime p_3 satisfy (2.14) . Thus $T_3 = \emptyset$.

2.15 Remark . In view of (2.2) and Lemma 2.13 , any n such that $n \in T_k$ for some $k \geq 5$ and $k \not\equiv 0 (\mathrm{mod}\ 4)$.

Now for any $k \geq 5$ let t_k be the unique integer such that $2^{t_k-1} < k \leq 2^{t_k}$. For example , $t_5 = t_6 = t_7 = t_8 = 3$; $t_9 = t_{10} = t_{11} = t_{12} = \cdots = t_{16} = 4$; ...and more generally $t_{2^j+1} = t_{2^j+2} = t_{2^j+3} = \cdots = t_{2^{j+1}} = j+1$ for $j \geq 2$.

An immediate consequence of the definition of t_k and the Lemma 2.4 is the following :

2.16 Lemma. If $n \in T_k$ with $\omega(n) = r$ then $r \geq t_k + 1$ or t_k according as n is even or odd .

2.17 Remark. It follows from Lemma 2.16 that any $n \in T_k$ $(k \geq 5)$ has at least t_k odd prime factors. In particular , in view Lemma 2.13 , it follows that any n has at least 3 odd prim factors.

The following result due Subbarao ([3], Theorem 2(B)) is used often in this paper without citing it also :

(2.18) If p_i and p_j are distinct odd prime factor of n then $p_i \not\equiv 1 (\mathrm{mod}\ p_j)$.

2.19 Lemma. Suppose k is odd, $n \in T_k$ and $\omega(n) = r$.Then $r - 3 < a_k, b_k,\ c_k$ or d_k according as $\gcd(n, 15) = 15, 3, 5$ or 1 respectively, where

$$a_k = \frac{(\ln 15k/32)}{\ln(32/17)}, \qquad b_k = \frac{(\ln 33k/80)}{\ln(32/17)}$$

$$c_k = \frac{(\ln 35k/96)}{\ln(24/13)} \quad and \quad d_k = \frac{(\ln 77k/240)}{\ln(24/13)}$$

Proof: Given k is odd, $n \in T_k$ with $\omega(n) = r$ .Then n is of the form (1.3) with $p_1 = 2$ and $p_i \not\equiv 1 (\mathrm{mod}\ p_j)$ for $2 \leq i \neq j \leq r$ and also $P_n = 2.\prod_{i=2}^{r}\left(\frac{p_i}{p_i-1}\right) = 2P_n^*$ (say) so that by (2.12) we have

(2.20) $\quad P_n^* > \frac{2^{r-1}}{k}$

i) Suppose $\gcd(n, 15) = 15$ so that $3 \mid n, and\ 5 \mid n$ .Hence

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$p_2 = 3, p_3 = 5, p_i \not\equiv 1(\mathrm{mod}\ 3), p_i \not\equiv 1(\mathrm{mod}\ 5)$ for $4 \le i \le r$ therefore

$p_i \ge 17$ for $i \ge 4$ and hence by (2.9),we get

(2.21) $P_n^* \le \frac{3}{2} \cdot \frac{5}{4} \cdot \left(\frac{17}{16}\right)^{r-3}$

Now combining (2.20) and (2.21) we get $\frac{2^{r-1}}{k} < \frac{15}{8} \cdot \left(\frac{17}{16}\right)^{r-3}$, which can be written as $\left(\frac{32}{17}\right)^{r-3} \cdot \frac{4}{k} < \frac{15}{8}$ or $\left(\frac{32}{17}\right)^{r-3} < \frac{15k}{32}$. giving the first part of the lemma.

ii) If $\gcd(n, 15) = 3$.then $p_2 = 3$ and by (2.18) , $p_3 \ge 11$ and $p_i \ge 17$ for $4 \le i \le r$ .hence $P_n^* \le \frac{3}{2} \cdot \frac{11}{10} \cdot \left(\frac{17}{16}\right)^{r-3}$ which together with (2.20) gives $\frac{2^{r-1}}{k} < \frac{33}{20} \cdot \left(\frac{17}{16}\right)^{r-3}$ and therefore $\left(\frac{32}{17}\right)^{r-3} < \frac{33k}{80}$ giving the second part of the Lemma .

iii) If $\gcd(n, 15) = 5$ then $p_2 = 5, p_3 \ge 7$ and $p_i \ge 13$ for $i \ge 4$, showing that $P_n^* \le \frac{5}{4} \cdot \frac{7}{6} \cdot \left(\frac{13}{12}\right)^{r-3}$. which together with(2.20) gives

$\left(\frac{24}{13}\right)^{r-3} < \frac{35\,k}{24}$ proving that $r - 3 < c_k$.

iv) If $\gcd(n, 15) = 1$ .then $p_2 \ge 7, p_3 \ge 11, p_i \ge 13$ for $i \ge 4$ .showing that $P_n^* \le \frac{7}{6} \cdot \frac{11}{10} \cdot \left(\frac{13}{12}\right)^{r-3}$ which together with(2.20) gives

$\frac{2^{r-1}}{k} < \frac{77}{60} \cdot \left(\frac{13}{12}\right)^{r-3}$ or $\left(\frac{24}{13}\right)^{r-3} < \frac{77\,k}{240}$ giving $r - 3 < d_k$

2.22 Lemma. Supposes k is even, $n \in T_k$ and $\omega(n) = r$.Then $r - 2 < a_k, b_k, c_k$ or d_k according as $\gcd(n, 15) = 15,3,5$ or 1 respectively where a_k, b_k, c_k, d_k are as defined in Lemma 2.19.

Proof: Given k is even, $n \in T_k$ and $\omega(n) = r$ .Then n is odd and if it is the form (1.3), then each p_i is odd

i) If $\gcd(n, 15) = 15$ then $p_1 = 3, p_2 = 5,$ and in view of (2.18) $p_i \ge 17$ for $i \ge 3$ so that by (2.10) and (2.12) , we get $\frac{2^r}{k} < \frac{3}{2} \cdot \frac{5}{4} \left(\frac{17}{16}\right)^{r-2}$ which can be written as $\left(\frac{32}{17}\right)^{r-2} < \frac{15k}{32}$ showing $r - 2 < a_k$.

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ii) If $\gcd(n,15)=3$ then $p_1=3, p_2 \geq 11$ and (2.18) , $p_i \geq 17$ for $i \geq 3$ so that $\frac{2^r}{k} < \frac{33}{20}.\left(\frac{17}{16}\right)^{r-2} \Rightarrow \left(\frac{32}{17}\right)^{r-2} < \frac{33k}{80} \Rightarrow r-2 < b_k$.

iii) If $\gcd(n,15)=5$ then $p_1=5, p_2 \geq 7$and $p_i \geq 13$ for $i \geq 3$ so that $\frac{2^r}{k} < \frac{35}{24}.\left(\frac{13}{12}\right)^{r-2}$ which implies $\left(\frac{24}{13}\right)^{r-2} < \frac{35\,k}{96} \Rightarrow r-2 < c_k$.

iv) If $\gcd(n,15)=1$. then $p_1 \geq 7, p_2 \geq 11$ and $p_i \geq 13$ for $i \geq 3$ so that $\frac{2^r}{k} < \frac{77}{60}.\left(\frac{13}{12}\right)^{r-2}$ which implies $\left(\frac{24}{13}\right)^{r-2} < \frac{77k}{240} \Rightarrow r-2 < d_k$.

Thus lemma is completely proved.

We give below a table of values of a_k, b_k, c_k and d_k for certain values of k.

k	a_k	b_k	c_k	d_k
5	1.35	1.14	0.97	0.77
6	1.63	1.433	1.27	1.06
7	1.88	1.67	1.528	1.31
9	2.27	2.07	1.93	1.72
10	2.44	2.24	2.10	1.90
11	2.59	2.39	2.265	2.05
13	2.85	2.65	2.537	2.32
14	2.97	2.77	2.65	2.45
15	3.08	2.88	2.77	2.56

(Table 1)

3. New proof of Theorem 1.5

In this part we present a proof of Theorem 1.5 which is entirely different from the one given in [3] .First we proof some lemmas .

3.1 Lemma. $T_5 = \emptyset$.

Proof : If possible, $n \in T_5$ and is of the form (1.3) . Then $p_1 = 2$ and since $t_5 = 3$. we get on one hand, by Lemma 2.16 Lemma that $r \geq 4$. On the other we have $r < a_5 + 3$, $b_5 + 3$,$c_5 + 3$ or $d_5 + 3$ according as $\gcd(n,15) = 15,3,5$ or 1 respectively , by Lemma 2.19, so that from Table 1, follows $r \leq 3$ if $\gcd(n,15) = 15$ or 1 ; and $r \leq 4$ in the case $\gcd(n,15) = 15$ or 3. Therefore $\gcd(n,15) = 5$ or 1 are impossible; and

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that $r = 4$ in the case $\gcd(n, 15) = 15$ or 3 showing $3 \mid n$ in both the cases .Thus if $n = p_1p_2p_3p_4$ then $p_1 = 2, p_2 = 3, 5 \leq p_3 < p_4$ and (2.3) with $k = 5$ and $r = 4$ gives

(3.2) $p_3p_4 + 17 = 16(p_3 + p_4)$.

Now we prove (3.2) in not solvable for primes p_3 and p_4 .

In case $\gcd(n, 15) = 15$ we have $p_3 = 5$ so that (3.2) reduces

$11p_4 + 63 = 0$ which is impossible for any prime p_4 .Therefore (3.2)is not solvable in this case. Also in case $\gcd(n, 15) = 3$ then $p_3 \not\equiv 1(\text{mod}3)$ and $p_4 \not\equiv 1(\text{mod}3)$ .Further if $p_3 \in \{q < 47: q\ is\ prime\ , q \not\equiv 1(mod\ 3)\}$ it is easy to see that there is no prime p_4 satisfying (3.2.) exists. That is for (3.2)to be solvable we must have $p_3 \geq 47$ and $p_4 \geq 53$. But in this case , by (2.10) and (2.9) , we get $3.2 = \frac{2^4}{5} < P_n \leq \frac{2}{1} . \frac{3}{2} . \frac{47}{46} . \frac{53}{52} = \frac{7473}{2392} = 3.12$ .a contradiction. Hence equation (3.2) is not solvable .Thus $T_5 = \emptyset$.

3.3 Lemma. If possible,$n \in T_6$ is in the form (1.3), Then by Lemma 2.16, we have on one hand $r \geq 3$; and on the other by Lemma 2.22 and table 1 , $r \leq 3$ .Thus $r = 3$ so that $n = p_1p_2p_3$ with $p_1 < p_2 < p_3$ where each p_i is odd and also

(3.4) $P_n > \frac{2^3}{6} = \frac{4}{3} > 1.33$

Also (2.3) with $k = 6$ and $r = 3$ gives

(3.5) $p_1p_2p_3 + 4(p_1 + p_2 + p_3) = 4(p_1p_2 + p_2p_3 + p_3p_1) + 3$.

Now we show that (3.5) is no solvable for odd primes p_1, p_2, p_3. First we prove $p_1 \notin \{3,5,7\}$.If $p_1 = 3$ then by (3.5) we have $p_2p_3 + 8(p_2 + p_3) = 9$ which is not solvable for primes p_2 and p_3 because the least value of the left greater than 9, if $p_1 = 5$ then (3.5) gives $p_2p_3 + 17 = 16(p_2 + p_3)$ for which we cannot find p_3 when $p_2 \in \{q < 37: q\ is\ \text{prime}\ , q \not\equiv 1(\text{mod}\ 5)\}$ that is means $p_2 \geq 37$ and $p_3 \geq 43$. In this case $P_n \leq \frac{5}{4} . \frac{37}{36} . \frac{43}{42} = \frac{7955}{6048} < 1.316$

Contradiction (3.4).

Finally If $p_1 = 7$ then (3.5) gives $p_2p_3 + 25 = 24(p_2 + p_3)$ there is no prime p_3 when $p_2 \in \{11,13\}$ showing $p_2 \geq 17$and hence $P_n \leq \frac{7}{6} . \frac{17}{16} . \frac{19}{18} = \frac{2261}{1728} < 1.309$, contradiction (3.4) . Thus $p_1 \geq 11$ so that $p_2 \geq 13$ in which

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case $P_n \leq \frac{11}{10}.\frac{13}{12}.\frac{17}{16} = \frac{2431}{1920} < 1.267$, again contradiction (3.4) .Hence (3.5) has no solution proving $T_6 = \emptyset$.

Notation. An equation will be said to be **Type *A*** if it not solvable in primes because the least value of one side of the equation greater than its other side .An equation to be solvable in primes will be said to be of **Type $\boldsymbol{B_q}$** if it is not solvable for the least among them less than the prim q.

For example in the proof of lemma 3.3 the equations reduce from (3.5) in the cases $p_1 = 3$, $p_1 = 5$ and $p_1 = 7$ are respectively equations Type A , B_{37} and B_{17} . As illustrated in the same lemma one can be prove that all equations of Type $\boldsymbol{B_q}$ are also not solvable .

3.6 Lemma . $T_7 = \emptyset$.

Proof : If possible, $n \in T_7$ and is the form (1.3) . Then using Lemma 2.16 , Lemma 2.19 and table 1 and by (2.20)

(3.7) $P_n^* > \frac{2^3}{7} > 1.142$.

Also if $n = p_1p_2p_3p_4$ with $2 = p_1 < p_2 < p_3 < p_4$,then (2.3) with $k = 7$ and $r = 4$ gives

(3.8) $p_2p_3p_4 + 8(p_2 + p_3 + p_4) = 8(p_2p_3 + p_3p_4 + p_2p_4) + 7$.

If $p_2 \in \{3,5,7,11,13,17,19\}$, from (3.8) we obtained the following equations and their type are given bellow :

p_2	**Reduced Equation**	**Type**
3	$5p_3p_4 + 16(p_3 + p_4) = 17$	A
5	$5p_3p_4 + 32(p_3 + p_4) = 33$	A
7	$p_3p_4 + 48(p_3 + p_4) = 49$	A
11	$3p_3p_4 + 81 = 80(p_3 + p_4)$	B_{71}
13	$5p_3p_4 + 97 = 96(p_3 + p_4)$	B_{47}
17	$9p_3p_4 + 129 = 128(p_3 + p_4)$	B_{31}
19	$11p_3p_4 + 145 = 144(p_3 + p_4)$	B_{31}

Table 2

That is for (3.8) to be solvable we must have $p_2 \geq 23$, $p_3 \geq 29$ and $p_4 \geq 31$ .But in this case

$P_n^* \leq \frac{23}{22}.\frac{29}{28}.\frac{31}{30} = \frac{20677}{18480} < 1.119$, a contradiction to (3.7) .that is $p_2 \geq 23$ is also impossible showing $T_7 = \emptyset$.

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Proof of Theorem 1.5

In view of (2.2) , Lemma 2.13, Lemma 3.1 , Lemma 3.3 , and Lemma 3.6 it follows that any composite $n > 4$ in T is in T_k for some $k \geq 9$ .since $t_k \geq 4$ for $k \geq 9$ we get by Remark 2.17, any composite $n > 4$ in T has at least four distinct odd prime factors.

4. Proof of Theorem 1.7.

Theorem 1.7 is an immediate consequence of lemma given bellow in view of Remark 2.17 , since $t_k \geq 5$ for $k \geq 17$.

1.1 Lemma. $T_k = \emptyset$ for $9 \leq k \leq 16$.

Proof : $T_{12} = T_{16} = \emptyset$, by (2.2).

We present the proof of $\boldsymbol{T_9} = \boldsymbol{T_{10}} = \emptyset$ only and we can use the same technique to proof the $T_{11} = T_{13} = T_{14} = T_{15} = \emptyset$.

i) If possible , $n \in T_9$ with $\omega(n) = r$ so that n is even and $r \geq 5$,by Lemma 2.16 on one hand , and on other by Lemma 2.19, and Table 1 , $r \leq 5$ if $\gcd(n, 15) = 15$ or 3 while $r \leq 4$ if $\gcd(n, 15) = 5$ or 1 .Therefore $\gcd(n, 15) = 5$ or 1 is impossible ; and that $r = 5$ if $gcd(n, 15) = 15$ or 3 .

Further if $n = p_1p_2p_3p_4p_5$ then $p_1 = 2$ and $p_2 = 3$ and $5 \leq p_3 \leq p_4 \leq p_5$ and (2.3) with $k = 9$, $r = 3$. gives

(4.2) $5p_3p_4p_5 + 32(p_2 + p_3 + p_4) = 32(p_3p_4 + p_3p_5 + p_4p_5) + 31$

Also (2.20) gives

(4.3) $P_n^* > \frac{2^4}{9} > 1.777$

Since $p_2 = 3$ it is follows $p_3 \notin \{7,13\}$. Putting $p_3 = 5$, $p_3 = 11$ successively in (4.2) the reduced equations are $128(p_4 + p_5) + 7p_4p_5 = 129$ and $23p_4p_5 + 321 = 320(p_4 + p_5)$ which are respectively type A and B_{23} . Therefore (4.2) is not solvable if $p_3 \in \{5,7,11,13\}$. Thus $p_3 \geq 17$ and hence $p_4 \geq 23$, $p_5 \geq 29$ in which case $P_n^* \leq \frac{3}{2} . \frac{17}{16} . \frac{23}{22} . \frac{29}{28} < 1.726$, a contradiction to (4.3) . That is , $p_3 \geq 17$ is also impossible shown $T_9 = \emptyset$.

ii) If possible, $n \in T_{10}$ and is of the form (1.3) . Then on one hand $r \geq 4$, by Lemma 2.16 ; and one the other by Lemma 2.22 and Table 1 , it is follows $r \leq 4$ in case $\gcd(n, 15) = 15{,}3$ or 5 while $r \leq 3$ in case

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$\gcd(n,15)=1$. Therefore $\gcd(n,15)\neq 1$. Also $r=4$ and $\gcd(n,15)=$ 15,3 or 5 . Further by (2.12) ,

4.4) $P_n^* > \frac{2^4}{10} = 1.6$

Let $n = p_1p_2p_3p_4$ with $p_1 < p_2 < p_3 < p_4$,where each p_i's is odd ; then (2.3) with $k=10$ and $r=4$ gives

(4.5) $3p_1p_2p_3p_4 + 8(p_1p_2 + p_1p_3 + p_1p_4 + p_2p_3 + p_2p_4 + p_3p_4) + 9$
$= 8(p_1p_2p_3 + p_1p_2p_4 + p_1p_3p_4 + p_2p_3p_4) + 8(p_1 + p_2 + p_3 + p_4)$

Now, we will show that (4.5) is not solvable for odd primes $p_i (i=1,2,3,4)$.

If $\gcd(n,15)=15$ then $p_1=3$, $p_2=5$ so that (4.5) reduces to $64(p_3+p_4)+11p_3p_4=65$,an equation of type A .Then $\gcd(n,15)\neq 15$.

If $\gcd(n,15)=3$ then $p_1=3$. we show that $p_2 \geq 47$. Clearly $p_2 \notin \{7,13,19,31,37,43\}$ since each prime q in the set is $\equiv 1(\text{mod}3)$. Also if $p_2 \in \{11,17,23,29,41\}$ then (4.5) reduce to the equations as given bellow :

p_2	Reduce Equation	Type
11	$5p_3p_4 + 160(p_3+p_4) = 161$	A
17	$p_3p_4 + 257 = 256(p_3+p_4)$	B_{509}
23	$7p_3p_4 + 353 = 352(p_3+p_4)$	B_{101}
29	$13p_3p_4 + 499 = 448(p_3+p_4)$	B_{71}
41	$25p_3p_4 + 641 = 640(p_3+p_4)$	B_{71}

Table 3

Thus $p_2 \geq 47$ so that $p_3 \geq 53$ and $p_4 \geq 59$ in which $P_n^* \leq \frac{3}{2}.\frac{47}{46}.\frac{53}{52}.\frac{59}{58} < 1.59$, a contradiction (4.4) . Hence $\gcd(n,15)\neq 3$.

Finally if $\gcd(n,15)=5$ then $p_1=5$. if $p_2=7$ then (4.5) reduce to $17p_3p_4 + 193 = 192(p_3+p_4)$, an equation of type B_{23} . Also $p_2 \neq 11$ since $11 \equiv 1(\text{mod}p_1)$. Therefore $p_2 \geq 13$, $p_3 \geq 17$ and $p_4 \geq 19$ in which case $P_n^* \leq \frac{5}{4}.\frac{13}{12}.\frac{17}{16}.\frac{19}{18} < 1.519$, a contradiction (4.4) . then $\gcd(n,15)\neq 5$. Thus $T_{10} = \emptyset$.

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Similarly we can show that $T_{11} = T_{13} = T_{14} = T_{15} = \emptyset$, complete the proof of Lemma 4.1

(4.6) Remark . As already noted the theorem 1.7 follows from Lemma 4.1 . Note that part (iii) of theorem 1.6 is improved , by showing $T_{15} = \emptyset$. Further using the method illustrated in this paper there is scope for improving Theorem 1.7 further .

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