On Set-Valued Mappings

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Abstract :

In this work, we discuss several properties and characterizations of continuous set-value mappings, we survey and discuss some weaker forms on this concept, the possibility of dependence of mathematical concepts on the concept of set-value mappings, as linearity, integration, differentiation and measurability etc, also we state and prove some theorems on certain types of set-value mappings.

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1- Introduction:

Weaker and stronger forms of points play an important role in analysis and topology, by using these points many authors introduced and studied various types of generalizations of sets and so as mappings, also; one of the important and basic topics in theory of classical point set topology and in several branches of mathematics, which has been investigated by many authors, is continuity of functions, this concept has been extended to the setting of set-value mappings, There are several weak and strong variants of continuity of set-value mappings in literature, for instance continuity [95, 127 and 160], strong continuity [6 and 101], super continuity [5], almost and weak continuity [112, 130-131 and 134], nearly and almost nearly continuity [36-37], semi-continuity [18 and 133], α-continuity, almost α-continuity and weak α-continuity [20, 140 and 142], precontinuity and almost precontinuity [154], quasi-continuity and almost quasi-continuity [88, 111, 113, 129, 135 and 141], γ-continuity and almost γ-continuity [3 and 40], δ-semicontinuity and δ-precontinuity [39 and 123], ℓ-continuity and almost ℓ-continuity [65], *c*-continuity and almost *c*-continuity [59 and 68], *c*-quasicontinuous [152] and *C-m*-continuous of set-value mapping [116], etc.

Moreover, a set-value mappings have many applications in applied mathematics and programming such as, optimal control, calculus of variation, probability, statistics, different inclusions, fixed-point theorems and even in economics, further; the original basic concept of functions represent an essential material for many of mathematical concepts, which the concepts of "Continuity, Differentiability, Integrability and Measurability etc", are begin with function, upbuilds and depends on function, thus functions play significant role in a subjects of mathematics, so the certain authors studied in good many of papers which have been extended much of these concepts to setting of set-value mappings, that; Y. S. Ledyaev and Q. J. Zhu, in 18 July 2006 introduce and study the concept of "Implicit Multifunction Theorems" [181], in Sep 2009, Chuji Wang, introduce and study the concept of "Fiber Loop Ringdown - a Time-Domain Sensing Technique for Multifunction Fiber Optic Sensor Platforms: Current Status and Design Perspectives" [176],

In Sep. 2002, EFE A. Ok, introduce and study the concept of "Functional Representation of Rotund-Valued Proper Multifunction" [117], in May-June 2002, E. J. Balder and A. R. Sambucini, introduce and study the concept of "On weak compactness and lower closure results for Pettis integrable multifunctions revision" [15].

In 2000, B. Cascales, V. Kadets and J. Rodrfgusz, introduce and study the concept of "Measurability And Selection Of Multifunction In Banach Spaces" [25], in 1991, S. Park, J. S. Bae, introduce and study the concept of "On zeros and fixed points of multifunctions with non-compact convex domains" [120], in 2009, Bozena Piatek, introduce and study the concept of "On The Continuity of Integrable Multifunction" [126], in 1995, C. Hess, introduce and study the concept of "On Measurability of Conjugate and Subdifferential of Normal Integrand" [55], in May 2003, E. J. Balder, introduce and study the concept of "Fatou's Lemma for Multifunctions with Unbounded Values in Dual Space" [16], in 2007, C. Z¸alinescu, A. I. Cuza and O. Mayer, introduce and study the concept of "HahnBanach Extension Theorems for Multifunctions revisited" [183], in 26-1-1972, H. Schirmer, introduce and study the concept of "Homotopy for Small Multifunction" [161], in 2008, Italy, E. Acerbi, G. Crippa and D.Mucci, introduce and study the concept of "A variational problem for couples of functions and multifunctions with interaction between leaves" [4], in 2004, Erdal Ekici, introduce and study the concept of" On some types of continuous fuzzy multifunction" [38], in 25 Oct 2004,C. Ursescu, introduce and study the concept of "Linear openess of multifunctions in metric spaces" [173], in 1993, S. Park, introduce and study the concept of "Fixed Point Theory Of Multifunctions In Topological Vector Spaces" [121], in 2004, J. Fiser, introduce and study the concept of "Numerical Aspects of Multivalued Fractals" [44], in 1991, D. Averna and G. Bonanno, introduce and study the concept of" Existence of solution for multivalued Boundary Value Poblem With Non-convex And Unbounded Right-Hand Side" [13], in 2001, D.Dentcheva, introduce and study the concept of "Approximation, Expansion and Univalued Representation of Multifunction" [34], in 1998, D. Dentcheva, introduce and study the concept of "Regular Castaing Representations Of Multimaps With Applications to Stochastic Programming" [33], in the left hand; the concepts of "Multifunctions And Graphs" and "Multifunctions with Closed Graphs", was study by J. E. Joseph and D. Holý, Trenčín [69 and 172],

Also the concepts of "Weaker form of B*-Continuity for Multimap" and "Upper and Lower NA-continuous Multifunctions" was introduced by D. K. Ganguly & C. Mitra, and by S. Yuksel, T. H. Simsekler and B. Kut [48], And the concept of "Integrability of multifunction" were introduced by R. J. Aumann, [12], a concept of "Differentiability of multifunction" were introduced by F. De Blasi, et al, [11, 31, 32, 74, 119, 159 and 171],…etc.

On the other hand; the concept of minimal structure which introduced by H. Maki in (1996) [84], as set-value mappings defined between two sets and satisfying certain minimal conditions, also in (2001) V. Popa and T. Noiri [149], introduced the concepts of "*m*-continuous functions" and "upper and lower *m*-almost continuous setvalued map".

Furthermore; the concept of continuous selection for multifunction which introduced by E. Michael [89-93] was represented a revolution in this area, where the many of mathematics applications could become to be as simulation to that selector single map of set-value map,

The concept of continuous selection, was good idea and useful beginning for studying the many of mathematical concepts, since the selector single-valued function can be represent as an approximation of set-valued map in the way of contraction or in the other ways, thus the mathematical concepts as; differentiation of set-valued maps which are crucial in many applications, and so that a typical set-valued map arising from some construction or variational problem will not be continuous, nonetheless; one often expects the maps to be outer semicontinuous, this however fails in some applications including generalized semi-infinite programming, thus there are a different notions of continuity of set-valued maps, which lead to notion of generalized differentiation of set-valued maps,

In this paper, we introduce and study certain types of continuous set-value map, and we investigate the relationships among another types for set-value mapping, also we give and discuss some studied applications for these types, and we will give some other proposed applications on these concepts. Our essential contribution, we investigate and study new application for set-value map on the concept of homotopy lifting property "H. L. P.", and some applications of known related concepts are also discussed.

In some books or papers a set-valued map from X to Y is denoted by $F:X\to 2^Y$, $F:X\rightsquigarrow Y$ or $F:X\rightrightarrows Y$, etc, but we exclusively use here the notation $F:X\to Y$,

Furthermore, the terms "set-valued map [11]), point-to-set map [56], correspondences [7], multivalued maps [61, 157-158], multifunction

[26], are usually used interchangeably; while the first being frequently used in current work "briefly; SV-map",

In this paper, the word spaces mean topological, and the capital letters F, H, G,… are denoted to a set-value mappings, and the small letters f, h, g,… are denoted to a single-mappings, and for a subset A of topological space (X, τ), $Cl(A)$ and $Int(A)$ represent the closure and interior of A with respect to τ, respectively.

We begin with the following terminologies and notions;

A subset A of X is said to be; α-open "resp. semi-open, preopen, β-open or semi-preopen, b-open or sp-open or γ-open", iff; $A\subset Int\{Cl[Int(A)]\}$ "resp. $A\subset Cl\{Int(A)\}$, $A\subset Int\{Cl(A)\}$, $A\subset Cl\{Int[Cl(A)]\}$, $A\subset Int\{Cl(A)\}\cup Cl\{Int(A)\}$", and for the details of all above concepts; see [1, 8-10, 24, 30, 35, 45, 50, 75-76, 82,84-87, 97-98, 102-106, 108-110, 122-123 and 150-151 and 155]

The family of all semi-open "resp. preopen, α-open, β-open, semi-preopen, b-open" sets in X is denoted by SO(X) "resp. $PO(X)$, α(X), β(X), SPO(X), BO(X)".

For these families, it is shown in [108-Lemm3.1] that $SO(X)\cap PO(X)=\alpha(X)$, since $\alpha(X)$ is a topology for X [103], by α-$Cl(A)$ "resp. α-$Int(A)$" we denote the closure "resp. interior" of A with respect to α(X), the complement of semi-open "resp. preopen, α-open" subsets is said to be semi-closed "resp. preclosed, α-closed",

The intersection of all semi-closed sets of X containing A is called semi-closure [30] of A and is denoted by $sCl(A)$, the union of all semiopen "resp. preopen" subsets of X contained in A is called the semi-interior "resp. preinterior" of A and is denoted by s-$Int(A)$ "resp. $preInt(A)$", a subset A of X is said to be regular-open "resp. regular closed" if $A=Int\{Cl(A)\}$ "resp. $A=Cl\{Int(A)\}$", the family of regular open "resp. regular closed" subsets of X is denoted by RO(X) "resp. RC(X)",

A subset E of X is said to be β-open [1], iff $E \subset Cl\{Int[Cl(E)]\}$, the family of all *β*-open subset of X is denoted by βO(X).

Also, by recall the definitions of θ-closure and δ-closure due to Velicko [174], that a point $x \in X$ is called θ-cluster "resp. δ-cluster" point of a subset $A \subset X$, iff $Cl(V) \cap A \neq \phi$ "resp. $Int\{Cl(V)\} \cap A \neq \phi$" for every open set V containing x, and a set of all θ-cluster "resp. δ-cluster" points of A is called θ-closure "resp. δ-closure" of A and is denoted by $Cl_\theta(A)$ "resp. $Cl_\delta(A)$" [174], a subset A is said to be θ-closed "resp. δ-closed) if $Cl_\theta(A)=A$ "resp. $Cl_\delta(A)=A$", the complement of θ-closed "resp. δ-closed" set is called θ-open "resp. δ-open".

The intersection of all *θ*-semiclosed sets "resp. semi-closed" of X containing A is called the *θ*-semiclosure [2] "resp. semiclosure [30]" of A and is denoted by $\theta\text{-}SCl(A)$ "resp. $SCl(A)$", the union of all *θ*-semiopen sets of X contained in A is called *θ*-semi-interior of A and is denoted by $\theta\text{-}SInt(A)$, so, a subset A of X is said to be;

- δ-semiopen "resp. θ-semiopen" [2, 39 and 122], iff $A \subset Cl\{Int_\delta(A)\}$ "resp. $A \subset Cl\{Int_\theta(A)\}$",
- δ-preopen "resp. θ-preopen" [35, 123 and 155], iff $A \subset Int\{Cl_\delta(A)\}$ "resp. $A \subset Int\{Cl_\theta(A)\}$",
- δ-*sp*-open "resp. θ-*sp*-open" [2 and 54], if $A \subset Cl\{Int[Cl_\delta(A)]\}$ "resp. $A \subset Cl\{Int[Cl_\theta(A)]\}$",

A collection of; δ-semiopen "resp. δ-preopen, δ-sp-open, θ-semiopen, θ-preopen, θ-sp-open" subsets of X are denoted by; δSO(X) "resp. δPO(X), δSPO(X), θSO(X), θPO(X), θSPO(X)", and these collections are all *m*-structures with property that "the union of any family of subsets belonging to m_X, also belongs to m_X", It is known that the families of all δ-open and θ-open sets of X are topologies for X.

Also, a subset $E\subset X$ is called α-paracompact "or strictly paracompact", [21, 67, 83 and 178] iff every cover of E by open sets of X is refined by a cover of E which consists of open sets of X and is locally finite in X.

For modifications of open sets defined above, the following relationships are known:

$$\begin{array}{ccccccccccccc}
\theta\text{-open} & \Rightarrow & \delta\text{-open} & \Rightarrow & \text{open} & \Rightarrow & \alpha\text{-open} & \Rightarrow & \text{preopen} & \Rightarrow & \delta\text{-preopen} & \Rightarrow & \theta\text{-preopen} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & & & \Downarrow & & \Downarrow \\
\theta\text{-semi-open} & \Rightarrow & \delta\text{-semi-open} & \Rightarrow & \text{semi-open} & \Rightarrow & b\text{-open} & \Rightarrow & sp\text{-open} & \Rightarrow & \delta\text{-}sp\text{-open} & \Rightarrow & \theta\text{-}sp\text{-open}
\end{array}$$

2- Preliminaries:

1-2) Definition: [11] ;

A SV-map $F : X \to Y$ is a point to set correspondence such that $F(x) \neq \phi$ for all $x\in X$, on the other hand, for each $x\in X$, there exist non-empty subset $F(x) \subset Y$,

The "Upper inverse" of a subset K, is define as $F^{+}(K)=\{x\in X : F(x)\subset K\}$, and the "Lower inverse" of a subset K, is define as $F^{-}(K)=\{x\in X : F(x)\cap K\neq\phi\}$,

A SV-map F:X$\to$Y is closed/open, iff an inverse image of any closed/open set is closed/open.

2-2) Definition:

A SV-map $F : X \to Y$ is called ;

- Upper continuous (U. C.) or Upper semi continuous (U. S. C.), iff for all $x \in X$, and for any open $V \subset Y$ contain $F(x)$, there is an open $U \subset X$ contain x, such that $F(U) \subset V$,
- Lower continuous (L. C.) or Lower semi continuous (L. S. C.), iff for any $x \in X$, and for any open $V \subset Y$ such that $F(x) \cap V \neq \phi$, there is open $U \subset X$ contain x, such that $F(U) \cap V \neq \phi$,
- Continuous "Semi Continuous" iff F has this property at each point of X.

Note: Some authors, defined the U. S. C., "resp. L. S. C." for SV-map F, as; iff $F^{+}(V)$ "resp. $F^{-}(V)$" is open in X, for any open set V in Y,

- Upper (or Lower) α-continuous "U./L. α-C. (or L. α-C.)" at $x_0 \in X$, iff for all open $V \subset Y$ contain $F(x_0)$ (or $F(x_0) \cap V \neq \phi$), there is an open $U \in \alpha O(X)$ contain x_0, such that $F(U) \subset V$ (or $F(u) \cap V \neq \phi$, for any $u \in U$).
- Upper/lower α-continuous (U/L. α-C.) at $x_0 \in X$, iff F has this property at all $x \in X$.
- Upper/lower almost continuous (U./L. A. C.), "resp. upper/lower almost α-continuous (U./L. A. α-C.), upper/lower almost quasi-continuous (U./L. A. q-C.), upper/lower almost pre-continuous (U./L. A. p-C.), upper/lower almost β-continuous (U./L. A. β-C.)", at $x_0 \in X$, iff for any open set $V \subset Y$ contain $F(x_0)$ or $F(x_0) \cap V \neq \phi$, there is an $U \in O(X, x)$ "resp. $U \in \alpha O(X, x)$, $U \in qO(X, x)$ $U \in PO(X, x)$

$U\in\beta O(X, x)$" contain x_0, such that $F(U)\subset Int\{Cl(V)\}$ or $F(u)\cap Int\{Cl(V)\}\neq\phi$ for all, $u\in U$,

- U/L. A. C., "resp. U/L. A. α-C., U/L. A. *q*-C.), U/L. A. *p*-C.), U/L. A. β-C.)", iff this property hold for each $x\in X$,
- Upper/Lower *b*-continuous U. *b*-C., "resp. L. *b*-C." at $x\in X$, if for all open $V\subset Y$ contain $F(x)$ "resp. $F(x)\cap V\neq\phi$", there is $U\in bO(X, x)$, s. t. $F(U)\subset V$ "resp. $F(u)\cap V\neq\phi$ for all $u\in U$",
- Upper/lower *b*-continuous (U/L. *b*-C.), iff *F* has this property at each point of X.
- Upper/Lower almost *b*-continuous "U/L. A. *b*-C." at $x\in X$, iff for all $V\subset Y$ with $x\in F^+(V)$ "or $x\in F^-(V)$", there is $U\in bO(X, x)$ with $U\subset x\in F^+\{Int[Cl(V)]\}$ "or $x\in F^-\{Int[Cl(V)]\}$",
- Upper/lower almost *b*-continuous (U/L. A. *b*-C.), iff *F* has this property at all $x\in X$.

For the SV-mapping defined above, the following implications hold:

$$\begin{array}{ccccc} \text{U. A. continuity} & \Rightarrow & \text{U. A. } \alpha\text{-continuity} & \Rightarrow & \text{U. A. pre-continuity} \\ & & \Downarrow & & \Downarrow \\ \text{U. A. quasi-continuity} & \Rightarrow & \text{U. A. } b\text{-continuity} & \Rightarrow & \text{U. A. } \beta\text{-continuity,} \end{array}$$

Note that none of these implications is reversible, so we give the following examples;

Let $X=Y=\{a, b, c, d\}$, and $\tau_X=\{\phi, \{a\}, \{b, c\}, \{a, b, c\}, X\}$, $\tau_Y=\{\phi, \{a\},\{b, d\},\{a, b, d\}, Y\}$, Define SV-map F :X $\to$ Y, by F(x)={x}, $\forall$ x, then F is upper almost β-continuous but not upper almost b-continuous, since {b, d}$\in$RO(Y) and F^{+}({b, d})={b, d} isn't b-open in τ_X.

Also, let X={a, b, c}, Y={1, 2, 3, 4, 5} and τ_X={ϕ,{a},{b},{a,b},X}, τ_Y={ϕ, {1, 2}, {3, 4}, {3, 4, 5}, {1, 2, 3, 4}, Y},

Define F : X $\to$ Y, by F(a)={1}, F(b)={3,4,5} and F(c)={2},

Then F is upper almost b-continuous but not upper almost precontinuous .

Also, let X ={1, 2, 3, 4, 5}, τ_X ={ϕ, {1, 2}, {3, 4}, {3, 4, 5}, {1, 2, 3, 4}, Y},

Define F :X$\to$Y, by F(1)={1}, F(2)={3}, F(3)={2}, F(4)={4}, and F(5)={5},

Then F is upper almost b-continuous but not upper almost quasi-continuous.

Also, a SV-map F : X $\to$ Y is called;

- Upper/Lower S-almost continuous (U./L. A. C. S.), at $x_0\in$X, iff for any open $V\subset$Y contain $F(x_0)$ "or; $F(x_0)\cap V\neq\phi$", there is open $U\subset$X contain x_0, such that $F(x)\subset Int\{Cl(V)\}$, "or; $F(x)\cap Int\{Cl(V)\}\neq\phi$", for each $x\in U$,
- S-almost continuous (A. C. S.), iff F has this property at each point of X.
- Upper/lower weakly continuous (U./L. W. C.), at x$\in$X, iff for any open V contain F(x), "or; $F(x_0)\cap V\neq\phi$" there is U contain x_0, with $F(x)\subset Cl(V)$, "or; $F(x)\cap Cl(V)\neq\phi$", for all $x\in U$",

- Weakly continuous (W. C.) at $x_0 \in X$, iff it has this property at each point of X.
- Upper almost weakly continuous (U. A. W. C.) at $x_0 \in X$, "resp. Lower almost weakly continuous (L. A. W. C.) at $x_0 \in X$", iff for any open $V \subset Y$ contain $F(x_0)$, then $x_0 \in Int\{Cl(F^{+}\{Cl(V)\})\}$, "resp. iff for each an open set $V \subset Y$ where $F(x_0) \cap V \neq \phi$, then $x_0 \in Int\{Cl(F^{-}\{Cl(V)\})\}$",
- Almost weakly continuous (A. W. C.) at $x_0 \in X$, iff F has this property at each $x \in X$.
- Upper quasi-continuous "resp. lower quasi-continuous", iff for any $x \in X$, all open $V \in Y$ containing $F(x)$, there is $U \in SO(X, x)$ such that $F(U) \subset V$, "resp. iff for any $x \in X$, and all open $V \in Y$ such that $F(x) \cap V \neq \phi$, there is $U \in SO(X, x)$, such that $F(u) \cap V \neq \phi$ for all $u \in U$".

For the above definitions, we put the following remarks and examples;

Reciprocally, if F is U. A. C. in x, it is obvious U. W. in x, as well,

If F is L. *S*-C. in x, it is obvious L. W. in x, as well,

The following implication holds;

- U. *S*-C $\Rightarrow$ U. A. C. $\Rightarrow$ U. W. C.
- L. *S*-C $\Rightarrow$ L. A. C. $\Rightarrow$ L. W. C.

The reciprocity is obvious, see [9, 11 and 20],

3-2) Definition :

A SV-map $F : X \to Y$ is called;

- Upper weakly continuous (U. W. C.) iff for each $x \in X$ and each open V containing $F(x)$, there exists an open set U containing x such that $F(U) \subset Cl(V)$,
- Upper weakly quasi continuous (U. W. q-C.) iff for all $x \in X$, any open U containing x, and any open V containing $F(x)$, there is a nonempty open G, where $G \subset U$ and $F(G) \subset Cl(V)$,
- Upper almost weakly continuous (U. A. W. C.) iff for each $x \in X$ and each open V containing $F(x)$, so that $x \in Int\{Cl(F^{+}\{Cl(V)\})\}$.
- Upper α-continuous (U. α-C.) at $x \in X$, iff for each open V containing $F(x)$, there exists $U \in \alpha(X, x)$ such that $F(U) \subset V$.
- Lower α-continuous (L. α-C.) at $x \in X$, iff for each open set V such that $F(x) \cap V \neq \phi$, there exists $U \in \alpha(X, x)$ such that $F(u) \cap V \neq \phi$, for every $u \in U$,
- Upper/Lower α-continuous, if it is upper (lower) α-continuous at all $x \in X$,
- Upper almost α-continuous (U. A. α-C.) at $x \in X$, iff for all $U \in SO(X, x)$ and all open V containing $F(x)$, there is a nonempty open $G \subset U$, such that $F(G) \subset Cl(V)$,
- Lower almost α-continuous (L. A. α-C.) at $x \in X$, iff for any $U \in SO(X, x)$, any open V such that $F(x) \cap V \neq \phi$, there is a nonempty open $G \subset U$ with $F(g) \cap sCl(V) \neq \phi$, for all $g \in G$,

- Upper/Lower almost α-continuous iff F has this property at every point of X.
- Upper weakly α-continuous (U. W. α-C.) at $x\in X$, iff for all $U\in SO(X, x)$ and any open V containing $F(x)$, there is nonempty open $G\subset U$, such that $F(G)\subset Cl(V)$,
- Lower weakly α-continuous (L. W. α-C.) at $x\in X$, iff for all $U\in SO(X, x)$, any open V with $F(x)\cap V\neq\phi$, there is a nonempty open $G\subset U$, such that $F(g)\cap Cl(V)\neq\phi$, for all $g\in G$,
- Upper/Lower weakly α-continuous, iff F has this property at every point of X,
- Weak* α-continuous iff for each open $V\subset Y$; $F^{-1}\{Fr(V)\}$ is α-closed, where $Fr(V)$ denotes the frontier of V,
- α-preopen if for every $U\in\alpha(X)$; $F(U)\subset Int\{Cl[F(U)]\}$.

For a SV-mapping defined above we have the following diagram;

$$\begin{array}{ccccc} & & \text{U. W. quasicontinuous} & & \\ & & \Uparrow & & \\ \text{U. }\alpha\text{-continuous} & \Rightarrow & \text{U. A. }\alpha\text{-continuous} & \Rightarrow & \text{U. W. }\alpha\text{-continuous} \\ & & \Downarrow & & \\ & & \text{U. A. W. continuous,} & & \end{array}$$

- Of course, every A. C. *S.* SV-map is W. C. SV-map, but the converse is not true in general, so we give the following example:

Let $X=\{a, b, c\}$, $\tau_X=\{\phi, \{a\}, \{a, b\}, X\}$ and $Y=\{1, 2, 3\}$, $\tau_Y=\{\phi, \{1\}, \{1, 2\}, Y\}$, by define $F:X\to Y$ as; $F(a)=\{1,3\}$, $F(b)=\{1,2\}$ and $F(c)=\{3\}$, so that F will be W.C., but not A.C.*S.*,

- Of course, every W. C. SV-map is A. W. C., but the converse is not true in general, so we give the following example:

Let X={a, b, c}, τ_X={ϕ, {a, b}, X} and Y={1, 2, 3}, τ_Y={ϕ, {2}, {1, 3}, Y}, We define F :X→Y as; F(a)={2}, F(b)=F(c)={1, 3}, so F will be A. W. C., but not W. C.,

For a SV-map F :X→Y, the graph SV-map G_F :X→X×Y is defined as G_F ={x}×F(x), x∈X, and a subset {{x}×F(x) : x∈X}⊂X×Y, is called a multigraph of F and is denoted by $G(F)$.

4-2) Definition:

A SV-map F : X → Y is called;

- Upper "or Lower) θ-semicontinuous, iff for any x∈X, any open V⊂Y such that x∈$F^+(V)$ "or x∈$F^-(V)$", there is θ-semiopen set U containing x such that $U\subset F^+(V)$. "or $U\subset F^-(V)$",
- U./L. A-θ-semicontinuous, iff for any x∈X, any open V⊂Y, with x∈$F^+(V)$ "or x∈$F^-(V)$", there is θ-semiopen U containing x, s. t. $U\subset F^+\{Int[Cl(V)]\}$, "or $U\subset F^-\{Int[Cl(V)]\}$",
- Upper/lower weakly θ-semicontinuous, iff for any x∈X, any open V with x∈$F^+(V)$, "resp. x∈$F^-(V)$", there is θ-semiopen U containing x with $U\subset F^+\{Cl(V)\}$, "resp. $U\subset F^-\{Cl(V)\}$",

In 1970, Gentry and Hoyle III [51] defined f :X→Y to be C-continuous at x∈X iff for any open V⊂Y contain f(x) and having compact complement, there is an open U⊂X containing x, with $f(U)\subset V$, some properties of C-continuous function studied by P. Long et al. [80-81 and 118], and in other papers, Neubrunn [100] and Hola et al. [59] extended this notion to the setting of SV-map, In 1991 Lipski [78], introduced the notion of C–quasicontinuous SV-map as generalizing of C-continuous and quasi-continuous SV-map, Some properties of C-quasi-continuous SV-map studied in [152].

In 2008, T. Noiri1 and V. Popa [116], introduced upper/lower C-m-continuous SV-map as SV-map defined on a set satisfying some minimal conditions, so they obtained some characterizations and several properties of such SV-map which turn out unify some results established in [59, 78 and 152], so that; a SV-map $F : X \to Y$ is called;

- Upper C-continuous (U. C. C.), "resp. upper C-quasicontinuous (U. C. q-C.)", iff for each open subset $V \subset Y$ contain $F(x)$ and having compact complement, there exist an open "resp. semi-open" subset $U \subset X$ contain x, such that $F(U) \subset V$,
- Lower C-continuous (L. C. C.), "resp. lower C-quasicontinuous (L. C. q-C.)" at $x \in X$, iff for each open set $V \subset Y$ meeting $F(x)$, and have compact complement, there exist an open "resp. semi-open" subset $U \subset X$ contain x, such that $F(u) \cap V \neq \phi$, for each $u \in U$,
- U./L. C-continuous "resp. U./L. C-quasi-continuous", iff F has this property at all $x \in X$.

For the SV-map defined above, the following implications hold:

"U. S. continuous $\Rightarrow$ U. C. continuous $\Rightarrow$ U. C. q-continuous";

"L. S. continuous $\Rightarrow$ L. C. continuous $\Rightarrow$ L. C. q-continuous",

Also, T. Noiri1 and V. Popa [116], defined the following modifications of upper/lower C-continuous SV-map, so that, a SV-map $F : X \to Y$ is called;

- Upper C-α-continuous "resp. upper C-precontinuous, upper C-b-continuous, upper C-sp-continuous" at $x \in X$, iff for all open V contain $F(x)$ and having compact complement, there is α-open "resp. preopen, b-open, semi-preopen" U contain x, such that $F(U) \subset V$,
- L. C-α-continuous "resp. L. C-precontinuous, L. C-b-continuous, L. C-sp-continuous" at $x \in X$, iff for any open $V \subset Y$ meeting $F(x)$, and have compact complement, there is α-open "resp. preopen, b-open, semi-preopen" $U \subset X$ contain x, s. t. $F(u) \cap V \neq \phi$, for each $u \in U$,
- Upper/Lower C-continuous "resp. upper/lower C-precontinuous, upper/lower C-b-continuous, upper/lower C-sp-continuous" if it has this property at each $x \in X$.

For SV-map defined above, the following relationships hold:

"upper semicontinuity $\Rightarrow$ upper α-continuity", and;

$$\begin{array}{ccccc} & & \text{Upper } C\text{-conts} \Rightarrow & \text{Upper } C\text{-}\alpha\text{-conts} \Rightarrow & \text{Upper } C\text{-precont s} \\ & & & \Downarrow & \Downarrow \\ & \text{Upper } C\text{-quasi-conts} \Rightarrow & & \text{Upper } C\text{-}b\text{-conts} \Rightarrow & \text{upper } C\text{-}sp\text{-conts} \end{array}$$

However, the converse implications are not true in general, and the analogous diagrams holds for the case of "lower".

There are several another types of continuity, so we can given further modifications by similar way, that a definition of any set, will be motivates to new types of these concepts, We enthrone this reviewing by the following modifications conclusions;

5-2) Definition:

A SV-map $F : X \to Y$ is called;

- Upper almost C-continuous (U. A. C-C.) "resp. Lower almost C-continuous (L. A. C-C.)" at $x\in X$, iff for any open V with $F(x)\subset V$, and has compact complement, there is open U with $x\in U$, such that $F(U)\subset V$, "resp. if for any open V with $F(x)\cap V\neq\phi$, and has compact complement, there exist an open $U\subset X$ contain x, such that $F(z)\cap V\neq\phi$, for each $z\in U$",
- Almost C-continuous at $x\in X$, iff it is both (U. A. C-C.) and (L. A. C-C.), at $x\in X$,
- Almost C-continuous, iff it is Almost C-continuous at each $x\in X$,
- Upper almost C-semicontinuous (U. A. C. S-C.) at $x\in X$, iff for any compact C with $F(x)\cap C=\phi$, there is open $U\subset X$ contain x, such that $F(z)\cap Cl\{Int(C)\}=\phi$ for $z\in U$,
- Lower almost C-semicontinuous (L. A. C. S-C.) at $x\in X$, iff whenever $Y\backslash V$ is compact and $F(x)\cap V\neq\phi$, there is open U contain x, such that $F(z)\cap Cl\{Int(V)\}\neq\phi$, for all $z\in U$,
- A. C. S-continuous at $x\in X$, iff it is both (U. A. C. S-C.) and (L. A. C. S-C.), at $x\in X$,
- Almost C. S-continuous, iff it is Almost C. S-continuous at each $x\in X$.

It is clear that; F is U./L. S-continuous, implies that F is U./L. A. C-continuous, and the following examples shows that these implications are not reversible in general, Let X and Y be the set of real R, u and τ be the usual and cofinite-topologies on X, Y resp.,

Define the SV-map F:X→Y; as follows:

$$F(\mathrm{x}) = \begin{cases} \{x\}, & \mathrm{x} \notin \{1,2,\ldots,n\} \\ \{1,2,\ldots,n\}, & \mathrm{x} \in \{1,2,\ldots,n\} \end{cases},$$

Then F is U. A. C-continuous, in fact F is A. C-continuous, but it is not U./L. S-continuous, because V=R\{1, 2,..., n}$\in\tau$, but not $F^+(V)$ nor $F^+(V)$ belongs to u,

Let X={a, b, c, d}, Y={1, 2, 3, 4} and τ_X={ϕ, {a}, {b}, {a, b}, {a, c, d}, X}, τ_Y={ϕ, {1}, {1, 2}, Y},

We define F :X→Y as; F(a)={4}, F(b)={1, 2}, F(c)={3} and F(d)={4},

So F will be U./L. S-continuous, and hence it is U./L. A. C-continuous, but it is not U./L. A. C. S-continuous at d, note that; C={1, 2, 3} is compact, and F(d)$\cap C$=ϕ, $Cl\{Int(C)\}$=Y, so that; there is no U contain d, such that F(x)$\cap Cl\{Int(C)\}$=ϕ for all x$\in U$.

6-2) Definition: see [116],

A SV-map F : X → Y is called;

- Upper C-θ-continuous "resp. upper C-θ-precontinuous, upper C-θ-semicontinuous, upper C-θ-sp-continuous" at x∈X, iff for any open Vcontain F(x) and V^C is compact, there is θ-open "resp. θ-preopen, θ-semiopen, θ-sp-open" U contain x, such that $F(U)\subset V$,
- L. C-θ-sp-continuous "resp. L. C-θ-precontinuous, L. C-θ-semicontinuous, L. C-θ-sp-continuous" at x∈X, iff for any open V

meeting F(x), and V^C is compact, there is θ-open "resp. θ-preopen, θ-semiopen, θ-*sp*-open" U contain x, such that $F(u)\cap V\neq\phi$, for all $u\in U$,

- U./L. C-θ-continuous "resp. U./L. C-θ-precontinuous, U./L. C-θ-semicontinuous, U./L. C-θ-*sp*-continuous", iff this property holds for each $x\in X$.
- U. C-δ-continuous "resp. U. C-δ-precontinuous, U. C-δ-semicontinuous, U. C-δ-*sp*-continuous" at $x\in X$, iff for all V contain F(x) and V^C is compact, there is δ-open "resp. δ-preopen, δ-semiopen, δ-*sp*-open" $U\subset X$ contain x, where $F(U)\subset V$,
- L. C-δ-*sp*-continuous "resp. L. C-δ-precontinuous, L. C-δ-semicontinuous, L. C-δ-*sp*-continuous" at $x\in X$, iff for any open V meeting F(x), and V^C is compact, there is δ-open "resp. δ-preopen, δ-semiopen, δ-*sp*-open" U contain x, such that $F(u)\cap V\neq\phi$, for all $u\in U$,
- U./L. C-δ-continuous "resp. U./L. C-δ-precontinuous, U./L. C-δ-semicontinuous, U./L. C-δ-*sp*-continuous", if it has this property for each $x\in X$.

For the SV-map defined above, the following relationships hold;

U. quasi-continuous$\Leftarrow$U. θ-semicontinuous$\Rightarrow$U. almost θ-semicontinuous$\Rightarrow$U. weakly θ-semicontinuous.

The following examples show that these implications are not reversible;

Let X={a, b, c}, τ_X ={ϕ, {a}, {c}, {a, c}, {b, c}, X},

Define F :X→X by; F(a)={a}, F(b)={b} and F(c)={c}, Then F is upper almost θ-semicontinuous, but not upper θ-semicontinuous,

Also, let X={a, b, c, d}, τ_X={ϕ, {a},{c}, {a, b}, {a, c}, {a, b, c}, {a, c, d},X},

Define F :X→X by; F(a)={a, b}, F(b)={d} and F(c)=F(d)={a, c, d},

Then F is upper weakly θ-semicontinuous, but not upper almost θ-semicontinuous,

And we have the following implications:

$$\begin{array}{cccccccccccc}
\textbf{U/L C.}\theta\textbf{-conts} & \Rightarrow & \textbf{U/L C.}\delta\textbf{-conts} & \Rightarrow & \textbf{U/L C.conts} & \Rightarrow & \textbf{U/L C.p.conts} & \Rightarrow & \textbf{U/L C.}\delta\textbf{-p.conts} & \Rightarrow & \textbf{U/L C.}\theta\textbf{-p.conts} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\textbf{U/L C.}\theta\textbf{-s.conts} & \Rightarrow & \textbf{U/L C.}\delta\textbf{-s.conts} & \Rightarrow & \textbf{U/L C.q.conts} & \Rightarrow & \textbf{U/L C.sp.conts} & \Rightarrow & \textbf{U/L C.}\delta\textbf{-sp.conts} & \Rightarrow & \textbf{U/L C.}\theta\textbf{-sp.conts}
\end{array}$$

In (1996), H. Maki [84], introduced the concept of minimal structure defined on a set, In (2001) V. Popa and T. Noiri [148], introduced the concept of m-continuous functions, And the concept of upper and lower m-almost continuous SV-map, also in [96] we studied the concept of m-continuous SV-map and we proved some result in this area.

So that we give the following definition,

7-2) Definition:

A subfamily m_X of power set P(X) of nonempty X, is called an minimal structure (briefly m-structure) on X, if $\phi \in m_X$, and $X \in m_X$, Each member of m_X is called m_X-open, and their complements is called m_X-closed, For any nonempty X, the pair (X, m_X) is called m-structure space, Let X is a nonempty, and m_X is an m-structure on X, for a subset A of X, the m_X-closure

and m_X-interior are defined as follows:

- $m_X\text{-}Cl(A)=\cap\{F : A \subset F \text{ and } X - F \in m_X\}$,

- m_X-Int(A)=$\cup\{U : U\subset$A and $U\in m_X\}$,

Note: Let (X, τ) be a topological space and A is any subset of X, if $\tau=m_X$, then;

$$m_X\text{-}Cl(A)=Cl(A) \text{ and } m_X\text{-}Int(A)=Int(A),$$

Let m_X be an m-structure on nonempty X, for a subsets A, B$\subset$X, the following are holding:

1) m_X -Cl(X - A)= X -{m_X - Int(A)}, and m_X-Int(X - A)= X -{m_X - Cl(A)},
2) If (X - A)$\in m_X$, then m_X -Cl(A)=A, and if A$\in m_X$, then m_X - Int(A)=A,
3) m_X -$Cl(\phi)=\phi$, m_X - Cl(X)=X, m_X-$Int(\phi)=\phi$ and m_X - Int(X)=X,
4) If A $\subset$ B, then m_X - Cl(A)$\subset m_X$ - Cl(B), and m_X - Int(A)$\subset$ m_X - Int(B),
5) If A $\subset$ m_X - Cl(A), and m_X - Int(A)$\subset$A,
6) m_X – Cl{m_X - Cl(A)}=m_X - Cl(A), and m_X - Int{m_X - Int(A)}=m_X - Int(A).

A minimal structure m_X on nonempty set X is said to be has property "ρ", if the union of any family of subsets are belong to m_X..,

A SV-map F :(X, m_X)$\rightarrow$(Y, m_Y) is calld;

- Upper m-continuous, iff for each x$\in$X and each $V\in m_Y$ containing F(x), there exist $U\in m_X$ containing x, such that $F(U)\subset V$,
- Lower m-continuous, iff for each x$\in$X and each $V\in m_Y$ such that $F(x)\cap V\neq\phi$, there exist $U\in m_X$ containing x, such that $F(u)\cap V\neq\phi$, for any $u\in U$.

Note: Let (X, τ_1) and (Y, τ_2) be a topological spaces, we put $\tau = m_X$, then; an upper (lower) m-continuous SV-map F :(X, m_X)$\rightarrow$(Y, τ_2) is an upper (lower) continuous SV-map.

Let X and Y be nonempty sets with minimal structure m_X, m_Y resp., an m-almost continuous SV-map, $F : (X, m_X) \rightarrow (Y, m_Y)$ is said to be;

- Upper m-almost continuous, iff for each $x \in X$ and any $V \in m_Y$ containing $F(x)$, there is $U \in m_X$ containing x, such that $F(U) \subset m_Y$ - $Int\{(m_Y - Cl(V)\}$,
- Lower m-almost continuous, iff for each $x \in X$ and $V \in m_Y$ such that $F(x) \cap V \neq \phi$, there is $U \in m_X$ containing x, where $F(u) \cap m_Y$ -$Int\{(m_Y - Cl(V)\} \neq \phi$, for any $u \in U$.

Of course, every m-continuous SV-map is m-Almost continuous, but the converse is not true in general, so we have the following example,

Let X={a, b, c}, m_X={ϕ, {a, b}, {c}, X} and Y={1, 2, 3}, m_Y={ϕ, {1}, {2}, {1, 2}, Y}, so a SV-map F :(X, m_X)$\rightarrow$(Y, m_Y), where F(a)={1, 2} and F(b)=F(c)={3}, is; M-almost continuous SV-map, but not m-continuous SV-map.

A minimal structure on space X is said to be an m-semiregular, iff for any $x \in X$ and m_X-open U containing x, there is an m_X-open V, such that $x \in V \subset m_X - Int\{(m_X - Cl(V)\} \subset U$,

Note: An m-space (X, m_X) is called m-regular, iff for any m_X-closed F, and $x \notin F$, have disjoint m_X-nbds, in other words, there is two m_X-open U, V, s. t. $F \subset U$, $y \in V$ and $U \cap V = \phi$, Let m_X, m_Y be a minimal structures on nonempty X, Y resp., if a SV-map F :(X, m_X)→(Y, m_Y) is m-Almost continuous, where Y is m-regular space, then F is m-continuous SV-map.

Before giving the next definition, we must point to the following notions,

- A cover of any space X by an open sets is said to be an open cover,
- A cover of any space X by an m_X-open sets is said to be an m_X-open cover.
- A set M in topological space X is called strictly m-paracompact iff every m_X-open cover for M in X can be refined by locally finite m_X-open cover in X.

So we nave the following definetions modifications;

8-2) Definition:

Let (X, m_X) be an m-space, a subset A of X is said to be; m_X-b-open iff;

$A \subseteq m_X - Cl\{m_X - Int(A)\} \cup \{m_X - Int[m_X - Cl(A)]\}$, the complement of an m_X-b-open set is called m_X-b-closed, the family of all m_X-b-open sets in (X, m_X) is denoted by m_X-bO(X).

In particular, the family of all m_X-b-open of (X, m_X) containing x∈X is denoted by m_X-bO(X, x), and the family of all m_X-b-closed in (X, m_X) is denoted by m_X-bC(X).

For the above definetions, we give the following remmarks and examples, If (X,m_X) is an m-space, then every m_X-open set is m_X-b-open, but an m_X-b-open set is not necessary to be m_X-open in general as shown in the following examples,

Let X={a, b, c} and m_X={ϕ, {a}, {b}, X}, so {a, b} is m_X-b-open but it is not m_X-open.

Let X={a, b, c, d, e} and m_X={ϕ, {a}, {e}, {c, d},X}, so {a, b, c} and {b, d, e} are m_X-b-open, but {a, b, c}∩{b, d, e}={b} is not m_X-b-open.

9-2) Definition:

A subset N_x of m-space (X, m_X) is said to be m_X-neighbourhood "resp. m_X-δ-nbd, m_X-δ-pre-nbd, m_X-b-nbd" of an point x∈X if there exsts a m_X-open "resp. m_X-δ-open, m_X-δ-preopen, m_X-b-open" set U, such that $x \in U \subseteq N_x$.

Let (X, m_X) be an m-space and a subset $A \subset X$, the m_X-b-closure of A, denoted by m_X-b-Cl(A), and the m_X-b-interior of A, denoted by m_X-b-Int(A), are defined, respectively, as;

- $m_X\text{-}b\text{-}Cl(A)=\cap\{U : X-U \in m_X\text{-}bO(X),\ A \subseteq U\}$,
- $m_X\text{-}b\text{-}Int(A)=\cup\{W : W \in m_X\text{-}bO(X),\ W \subseteq A\}$,

Also, we need to the following definition;

10-2) Definition: see [125 and 175],

A subset H of m-space (X, m_X) is said to be m_X-regular open "resp. m_X-preopen", iff $H= m_X\text{-}Int(m_X\text{-}Cl(H))$ "resp. $H \subseteq m_X\text{–}Int\{m_X\text{-}Cl(H)\}$", the complement of an m_X-regular open "resp. m_X-preopen" set is said to be m_X-regular closed "resp. m_X-preclosed",

Let A be a subset of m-space (X, m_X);

- The intersection of all m_X-preclosed sets of X containing A is called m_X-preclosure of A and is denoted by m_X-pCl(A), and the union of all m_X-preopen sets of X contained in A is called m_X-preinterior of A and is denoted by m_X-$pInt$(A).

Let A be a subset of m-space (X, m_X);

- The union of all m_X-regular open sets of X contained in A is called m_X-δ-interior of A and is denoted by m_X-δInt(A).
- A subset A⊂X is called δ-open, iff A=m_X-δInt(A), the complement of δ-open is δ-closed.

A subset H of m-space (X, m_X) is called m_X-δ-preopen, iff H⊆ m_X-Int{m_X-δCl(H)}, the complement of m_X-δ-preopen is called m_X-δ-preclosed.

Let A be a subset of m-space (X, m_X);

- The intersection of all m_X-δ-preclosed of X containing A is called m_X-δ-preclosure of A and is denoted by m_X-δ-pCl(A).
- The union of all m_X-δ-preopen of X contained in A is called m_X-δ-preinterior of A and is denoted by mX-δ-pInt(A).

Let F :(X, m_X)→(Y, σ), of m-space (X, m_X) to topological space (Y, σ), so F is said to be:

- Upper δ-m-precontinuous at x∈X, iff for each open V containing F(x), there exists an m_X-δ-preopen set U containing x, such that F(U)⊆V,

- Lower δ-m-precontinuous at $x\in X$ iff for each open V such that $F(x)\cap V\neq\phi$, there exists an m_X-δ-preopen U containing x, such that $F(z)\cap V\neq\phi$ for every $z\in U$,
- Upper/Lower δ-m-precontinuous, iff F has this property at each point of X.
- Upper almost δ-m-precontinuous at $x\in X$, iff for each open V, such that $x\in F^{+}(V)$, there exists an m_X-δ-preopen U containing x, such that $U\subseteq F^{+}\{Int[Cl(V)]\}$,
- Lower almost δ-m-precontinuous at $x\in X$, iff for each open V, such that $x\in F^{-}(V)$, there exists an m_X-δ-preopen U containing x, such that $U\subseteq F^{-}\{Int[Cl(V)]\}$,
- Upper/Lower almost δ-m-precontinuous, iff F has this property at all point of X.

A SV-map $F:(X, m_X)\rightarrow(Y, \sigma)$, of m-space (X, m_X) into topological space (Y, σ), is called;

- Upper b-M-continuous at $x\in X$, iff for any open $V\in Y$ such that $F(x)\subseteq V$, there exist $U\in m_X$-$bO(X, x)$ such that $F(U)\subseteq V$,
- Lower b-M-continuous at $x\in X$, iff for each open $V\in Y$ such that $F(x)\cap V\neq\phi$, there exists $U\in m_X$-$bO(X, x)$ such that $F(u)\cap V\neq\phi$, for every $u\in U$,
- And F is called upper/lower b-M-continuous if F is upper/lower b-M-continuous for all $x\in X$.

For the above definetios, we give the following examples;

Let X={a, b, c} and Y={1, 2}, define m_X={ϕ, {a}, {b}, {a, c}, {b, c}, X}, and σ_Y ={ϕ, {1}, Y}, so a SV-map F :(X, m_X)→(Y, σ) which defined as; F(a)=F(b)={2} and F(c)=Y, is to be upper *b*-*M*-continuous.

Let X={a, b, c} and Y={1, 2}, define minimal structure m_X={ϕ, {a}, {b}, {a, c}, {b, c}, X} on X, and a topology σ_Y ={ϕ, {1}, Y} on Y, so a SV-map F : (X, m_X)→(Y, σ) which defined as; F(a)=F(b)=Y and F(c)={1}, is to be lower *b*-*M*-continuous.

Also, a SV-map F :(X, m_X)→(Y, σ), is said to be;

- Upper almost *b*-*M*-continuous at x∈X, iff for each open set V∈Y such that x⊆F^+(V),
  there exists U∈m_X-bO(X, x) such that U⊆F^+{Int[Cl(V)]}, and F is called upper almost
  b-*M*-continuous if F is upper almost *b*-*M*-continuous for all x∈X.
- Lower almost *b*-*M*-continuous at x∈X, iff for each open V∈Y such that x⊆F^-(V), there exists U∈ m_X-bO(X, x) such that U⊆F^-{Int[Cl(V)]}, and F is called lower almost *b*-*M*-continuous if F is lower almost *b*-*M*-continuous for all x∈X.

For the above definetios, we give the following remmarks and examples;

"U. *b*-*M*-continuous ⇒ U. A. *b*-*M*-continuous",

But this implication is reversible, so we give the following example;

Let X={a, b, c, d} and Y={1, 2, 3}, with m_X ={∅, {c}, {d}, X}, and σ_Y ={∅, {1}, Y}, so a SV-map F :(X, m_X)→(Y, σ) which defined as;

F(a)={1}, F(b)={1, 2} and F(c)=F(d)=Y, is to be upper almost b-M-continuous, but it is not upper b-M-continuous.

Also, we have that; "L. b-M-continuous ⇒ L. A. b-M-continuous",

But this implication is reversible, so we give the following example;

Let X={a, b, c, d} and Y={1, 2, 3}, with m_X ={∅, {c}, {d}, X}, and σ_Y ={∅, {1}, Y}, so a SV-map F :(X, m_X)→(Y, σ) which defined as; F(a)={1}, F(b)=Y and F(c)=F(d)={2, 3}, is to be lower almost b-M-continuous but it is not lower b-M-continuous.

For a SV-map F, defined above, the following implication hold:

"Upper δ-m-precontinuity ⇒ Upper almost δ-m-precontinuity"

Note that none of these implication is reversible, so we give the following example;

Let X={a, b, c} and Y={1, 2, 3, 4, 5}, with m_X ={ϕ, {b}, {c}, {b, c}, X} and τ_Y ={ϕ, {1, 2, 3, 4}, Y}, define F :(X, m_X)→(Y, σ), by F(a)={3}, F(b)={2, 4} and F(c)={1, 5}, then F is U. A. δ-m-pre-continuous, but not U. δ-m-precontinuous, since {1, 2, 3, 4}∈τ_Y, and F^+({1, 2, 3, 4, 5})= {a, b} is not m_X-δ-preopen in X.

3- Main results:

In this section, we discuss and prove some results on the concept of U/L. δ-m-pre-continuous, we begin with the following four theorems which due to the E. Ekici [40], that he collect many of important basic terms for the generalize forms of SV-map.

1-3) Theorem: [39],

Let F : X→Y be a SV-map, then the following statements are equivalent;

1- F is U. A. δ-semicontinuous SV-map,

2- $F^{+}\{Int[Cl(V)]\}\in\delta\text{-}SO(X)$, for any open $V\subset Y$,

3- $F^{-}\{Cl[Int(K)]\}\in\delta\text{-}SC(X)$, for any closed $K\subset Y$,

4- $F^{+}(G)\in\delta\text{-}SO(X)$, for any regular open $G\subset Y$,

5- $F^{-}(E)\in\delta\text{-}SC(X)$, for any regular closed $E\subset Y$,

6- For all $x\in X$, any open V and $F(x)\subset V$, there is δ-semiopen U and $x\in U$, such that $F(U)\subset S.Cl(V)$,

7- $F^{+}(V)\subset\delta\text{-}S.Int\{F^{+}\{S.Cl(V)\}$, for all open $V\subset Y$,

8- $\delta\text{-}S.Cl\{F^{-}[S.Int(K)]\}\subset F^{-}(K)$, for all closed $K\subset Y$,

9- $\delta\text{-}S.Cl\{F^{-}(Cl[Int(K)])\}\subset F^{-}(K)$, for all closed $K\subset Y$,

10- $\delta\text{-}S.Cl\{F^{-}(V)\}\subset F^{-}\{Cl(V)\}$, for each $V\in bO(Y)$,

11- $\delta\text{-}S.Cl\{F^{-}(V)\}\subset F^{-}\{Cl(V)\}$, for each $V\in S.O(Y)$,

12- $F^{+}(V)\subset\delta\text{-}S.Int\{F^{+}\{Int[Cl(V)]\}$, for every $V\in pO(Y)$,

2-3) Theorem: [39],

Let F:X→Y be a SV-map, then the following statements are equivalent;

1- F is L. A. δ-semicontinuous SV-map,

2- $F^{-}\{Int[Cl(V)]\}\in\delta\text{-}SO(X)$, for any open $V\subset Y$,

3- $F^{+}\{Cl[Int(K)]\}\in\delta\text{-}SC(X)$, for any closed $K\subset Y$,

4- $F^{-}(G)\in\delta\text{-}SO(X)$, for any regular open $G\subset Y$,

5- $F^{+}(E)\in\delta\text{-}SC(X)$, for any regular closed $E\subset Y$, 6- For all $x \in X$, any open V with $F(x) \cap V \neq \phi$, there is δ-S.open U and $x \in U$, s. t. $F(u) \cap S.Cl(V) \neq \phi$,

7- $F^{-}(V) \subset \delta\text{-}S.Int\{F^{-}\{S.Cl(V)\}$, for all open $V \subset Y$,

8- $\delta\text{-}S.Cl\{F^{+}[S.Int(K)]\} \subset F^{+}(K)$, for all closed $K \subset Y$,

9- $\delta\text{-}S.Cl\{F^{+}(Cl[Int(K)])\} \subset F^{+}(K)$, for all closed $K \subset Y$,

10- $\delta\text{-}S.Cl\{F^{+}(V)\} \subset F^{+}\{Cl(V)\}$, for each $V \in bO(Y)$,

11- $\delta\text{-}S.Cl\{F^{+}(V)\} \subset F^{+}\{Cl(V)\}$, for each $V \in S.O(Y)$,

12- $F^{-}(V) \subset \delta\text{-}S.Int\{F^{-}\{Int[Cl(V)]\}$, for every $V \in pO(Y)$,

3-3) Theorem: [39],

Let $F : X \to Y$ be a SV-map, then the following statements are equivalent;

1- F is U. W. δ-semicontinuous SV-map,

2- For each $x \in X$ and each open V containing $F(x)$, there exists an δ-semiopen U containing x, such that $F(U) \subset Cl(V)$,

3- $F^{+}(V) \subset Cl\{\delta\text{-}Int(F^{+}[Cl(V)])\}$, for any open $V \subset Y$,

4- $Int\{\delta\text{-}Cl[F^{-}(V)]\} \subset F^{-}\{Cl(V)\}$, for any open $V \subset Y$,

5- $Int\{\delta\text{-}Cl(F^{-}[Int(K)]\} \subset F^{-}(K)$, for any closed $K \subset Y$,

6- $\delta\text{-}S.Cl\{F^{-}[Int(K)]\} \subset F^{-}(K)$, for any closed $K \subset Y$,

7- $\delta\text{-}S.Cl\{F^{-}(Int[Cl(E)])\} \subset F^{-}\{Cl(E)\}$, for any subset $E \subset Y$,

8- $F^{+}\{Int(E)\} \subset \delta\text{-}S.Int\{F^{+}(Cl[Int(E)])\}$, for subset $E \subset Y$,

9- $F^{+}(V) \subset \delta\text{-}S.Int\{F^{+}[Cl(V)]\}$, for any open $V \subset Y$,

10- $\delta\text{-}S.Cl\{F^{-}(V)\} \subset F^{-}\{Cl(V)\}$, for any open $V \subset Y$.

4-3) Theorem: [39],

Let F : X→Y be a SV-map, then the following statements are equivalent;

1- F is L. W. δ-semicontinuous SV-map,

2- For each x∈X and each open V such that $F(x)\cap V\neq\phi$, there exists an δ-semiopen U containing x, such that if y∈U, then $F(y)\cap Cl(V)\neq\phi$,

3- $F^{-}(V)\subset Cl\{\delta\text{-}Int(F^{-}[Cl(V)])\}$, for any open $V\subset Y$,

4- $Int\{\delta\text{-}Cl[F^{+}(V)]\}\subset F^{+}\{Cl(V)\}$, for any open $V\subset Y$,

5- $Int\{\delta\text{-}Cl(F^{+}[Int(K)]\}\subset F^{+}(K)$, for any closed $K\subset Y$,

6- $\delta\text{-}S.Cl\{F^{+}[Int(K)]\}\subset F^{+}(K)$, for any closed $K\subset Y$,

7- $\delta\text{-}S.Cl\{F^{+}(Int[Cl(E)])\}\subset F^{+}\{Cl(E)\}$, for any subset $E\subset Y$,

8- $F^{-}\{Int(E)\}\subset\delta\text{-}S.Int\{F^{-}(Cl[Int(E)])\}$, for subset $E\subset Y$,

9- $F^{-}(V)\subset\delta\text{-}S.Int\{F^{-}[Cl(V)]\}$, for any open $V\subset Y$,

10- $\delta\text{-}S.Cl\{F^{+}(V)\}\subset F^{+}\{Cl(V)\}$, for any open $V\subset Y$,

In the end of this section we state and prove the following four theorems, but in beigen we need to the following lemma,

5-3) Lemma: [112],

For a SV-map F :X→Y, and any subsets A⊂X, B⊂Y, the following assertions hold:

1- $G_F^{+}(A\times B)=A\cap F^{+}(B)$,

2- $G_F^{-}(A\times B)=A\cap F^{-}(B)$.

6-3) Theorem:

Let $F:X\to\prod_{i\in I}X_i$ be SV-map from topological space X to product space $\prod_{i\in I}X_i$ and let $P_i:\prod_{i\in I}X_i\to X_i$ be the projection for all $i\in I$, so if F is U./L. δ-m-precontinuous, then $P_i\circ F$ is U./L. δ-m-precontinuous SV-map for each $i\in I$.

Proof:

We shall prove this only for the upper case, and the lower case is similar,

Let V_i be an open in X_i, since $P_i(V_i\times\prod_{i\in I}X_i)$ is open, so take

$\{P_i\circ F\}(x)\subset P_i(V_i\times\prod_{i\in I}X_i)$,

Since $\{P_i\circ F\}(x)=P_i\{F(x)\}$, and F is U. δ-m-p-continuous, also P_i is continuous,

So that; there exists an m_X-δ-preopen set U containing x, such that $F(U)\subset V_i\times\prod_{i\in I}X_i$,

And hence; $\{P_i\circ F\}(U)\subset P_i(V_i\times\prod_{i\in I}X_i)$, then $F\circ P_i$ is U. δ-m-precontinuous SV-map.

7-3) Theorem:

Let $F:X\to Y$ be maultifunction, and E be an m_X-δ-open set in X, if F is U/L. δ-m-precontinuous, so the restriction SV-map $F|E:E\to Y$ is U./L. δ-m-precontinuous.

Proof:

Suppose that V is an open in Y, let $x\in E$ and $F(x)\subset V$,

Since F is U. δ-m-precontinuous, it follows that there exists an m_X-δ-preopen G, where $x\in G$ and $F(G)\subset V$, so that $x\in G\cap E\in\delta\text{-}m_X\text{-}pO(E)$, and $\{F|E\}(G\cap E)\subset V$,

Thus, we show that the restriction SV-map $F|E$ is U. δ-*m*-precontinuous,

The proof of the lower case is similar to that given above.

8-3) Theorem:

Let $F:X\to Y$ be maultifunction, if the graph SV-map of F is U/L. δ-*m*-precontinuous, then F is U/L. δ-*m*-precontinuous.

Proof:

Suppose that $G_F:X\to Y\times Y$ is U. δ-*m*-precontinuous, $x\in X$ and V be any open of Y containing $F(x)$, Since $X\times V$ is open in $X\times Y$ and $G_F(x)\subset X\times V$, there is $U\in m_X\text{-}\delta\text{-}pO(X,x)$ such that $G_F(U)\subset X\times V$, and we have $U\subset G_F^+(X\times V)=X\cap F^+(V)=F^+(V)$ "by Lemma 5-3",

So $F(U)\subset V$, which showes that F is U/L. δ-*m*-precontinuous.

The proof of the lower case is similar to that given above.

9-3) Theorem:

Suppose that $F_1:X\to Y$ and $F_2:X\to Z$ are SV-map, let $F_1\times F_2:X\to Y\times Z$ be a SV-map which defined by $\{F_1\times F_2\}(x)=F_1(x)\times F_2(x)$, for all x, if $F_1\times F_2$ is U/L. W. δ-precontinuous, then F_1 and F_2 is U/L. W. δ-precontinuous.

Proof:

Let $x\in X$ and V_1, V_2 be any open sets of Y, Z resp., with $x\in F_1^+(V_1)$ and $x\in F_2^+(V_2)$,

Such that; $F_1(x)\subset V_1$, $F_2(x)\subset V_2$, hence $F_1(x)\times F_2(x)=\{F_1\times F_2\}(x)\subset V_1\times V_2$, And thus; $x \in \{F_1 \times F_2\}^+ (V_1 \times V_2)$, it follows that there exists δ-preopen U containing x such that $U \subset \{F_1 \times F_2\}^+ [Cl(V_1 \times V_2)]$, we obtain that $U \subset F_1^+\{Cl(V_1)\}$ and $U \subset F_2^+\{Cl(V_2)\}$,

Therefore F_1 and F_2 is U. W. δ-precontinuous.

The proof of the lower case is similar to that presented above.

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